**Numerical Analysis Preliminary Examination, May 2017**

Work **7** of the following 8 problems. If you attempt all 8 problems, clearly indicate which 7 problems are to be graded. If not, problems 1 through 7 will be graded.

1. Prove that an $n \times n$ real matrix $A$ has a Cholesky factorization if and only if $A$ is symmetric positive definite.

2. Let $A$ be the $2 \times 2$ matrix

$$\begin{bmatrix} a & -b \\ -2a & a \end{bmatrix},$$

where $a$ and $b$ are positive real numbers.

   (a) Find all values of $b/a$ such that Jacobi iteration is convergent.

   (b) Find all values of $b/a$ such that Gauss-Seidel iteration is convergent.

3. Consider the linear system $Ax = b$ where $A$ is a nonsingular $n \times n$ matrix. Let $\Delta A$ be a perturbation of $A$ satisfying $\|\Delta A\| \|A^{-1}\| < 1$. Prove that if $\Delta x$ satisfies

$$(A + \Delta A)(x + \Delta x) = b,$$

then

$$\frac{\|\Delta x\|}{\|x\|} \leq \frac{\|A\| \|A^{-1}\|}{1 - \|\Delta A\| \|A^{-1}\|} \frac{\|\Delta A\|}{\|A\|}.$$

4. Consider an iteration function $g(x)$ of the form $g(x) = x - f(x)f'(x)$. (Note: This is NOT a Newton iteration function.) Assume that $r$ satisfies $f(r) = 0$ and $f'(r) \neq 0$. Find the precise conditions on the function $f$ so that the iterations $x_{n+1} = g(x_n)$ converge to the fixed-point $r$ at least *cubically* if started near $r$.

5. Consider integrals of the form

$$I(f) \;=\; \int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}}\, dx.$$

   (a) Derive nodes and weights for a Gaussian quadrature rule $Q$ that computes $I(f)$ exactly for all $f \in P^3$.

   (b) Derive a bound on the error $|I(f) - Q(f)|$ in terms of $\|f^{(k)}\|_\infty$ for some $k$, which you should specify.

6. Let $V = C[-1,1]$ be the space of continuous functions on $[-1,1]$. Let $\|\cdot\|_2$ be the $L^2$ norm associated with the inner product $(f,g) = \int_{-1}^{1} f(x)g(x)dx$.

   (a) Let $W$ be a finite dimensional subspace of $V$. State what it means for a function $w \in W$ to be the *best approximation* to $f \in V$ from the space $W$ with respect to the norm $\|\cdot\|_2$.

   (b) Find the first-degree polynomial that best approximates $f(x) = 4x^3 - 1$ in the norm $\|\cdot\|_2$.

   (c) Let $p_n \in P^n$ be the least squares approximation to a function $f \in C[-1,1]$. Prove that $(f - p_n, q_n) = 0$ for any $q_n \in P^n$.

7. Consider *Heun's method* for the scalar IVP $y' = f(x,y)$, $y(0) = y_0$.

$$\begin{aligned} x_{n+1} &= x_n + h \\ K_1 &= f(x_n, y_n) \\ K_2 &= f(x_{n+1}, y_n + hK_1) \\ y_{n+1} &= y_n + \frac{h}{2}\left[K_1 + K_2\right]. \end{aligned}$$

   (a) Prove that given certain assumptions about the differentiability of $f$, the local truncation order is $O(h^3)$. What are the least differentiability assumptions required for this error estimate to hold?

   (b) State the definitions of absolute stability and $A$-stability.

   (c) Find the real part of the stability region (domain of absolute stability) for this method.

8. Consider the nonlinear boundary value problem,

$$-u'' = \frac{1}{2}(1 + \sin(u)),$$

   posed on $(0,1)$ with boundary conditions $u(0) = u(1) = 0$.

   (a) Write down the nonlinear algebraic system of equations resulting from the finite difference method with $N$ internal nodes.

   (b) Consider the iterative strategy of solving the nonlinear system with an initial solution vector $u^{(0)}$ and iterating $Au^{(n+1)} = F(u^{(n)})$, where $A$ is the finite difference matrix obtained by discretizing $-u''$ and $F(u^{(n)})_i = \frac{1}{2}(1 + \sin u_i^{(n)})$. Show that this iteration converges to the solution of the algebraic equation for any initial input. (Hint: the fact that the eigenvalues of $A$ are known to be $\{2N^2\left(1 - \cos\left(\frac{\pi j}{N+1}\right)\right)\}_{j=1}^{N}$ may be helpful in determining the norm of $A^{-1}$ and/or $A$.)