Numerical Analysis Preliminary Examination, August 2017

Work **8** of the following 9 problems. If you attempt all 9 problems, clearly indicate which 8 problems are to be graded. If not, problems 1 through 8 will be graded.

1. Let $x \in \mathbb{R}^n$ with $x = [x_1, x_2, x_3, \ldots, x_n]^T$ and $x_1 \neq 0$. Let $u = x + \sigma e_1$, where $\sigma = \operatorname{sign}(x_1)\|x\|_2$, and let $\theta = \frac{1}{2}\|u\|_2^2$. Finally, let $U = I - \frac{1}{\theta}uu^T$. Prove that U is unitary and that $Ux = -\sigma e_1$.

2. Let $A \in \mathcal{R}^{n\times n}$.

   (a) Prove that the trace of A equals the sum of its eigenvalues.

   (b) Prove that if the eigenvalues of A satisfy $|\lambda_1| > |\lambda_i|$ for $i = 2, 3, \ldots, n$, then

   $$\lambda_1 = \lim_{m\to\infty} \frac{tr\,(A^{m+1})}{tr\,(A^m)}.$$

3. Let A be an $n\times n$ matrix and let $Q = L+D$ be the lower triangular part of A, including the diagonal. Prove that if A is strictly diagonally dominant, the Gauss-Seidel method

   $$Qx^{(k+1)} = (Q - A)\,x^{(k)} + b, \quad k = 0, 1, 2, \ldots$$

   converges to the solution of $Ax = b$ for any starting vector $x^{(0)}$.

4. Consider the equation $x^3 - x - 1 = 0$ which has a root ξ between 1 and 2.

   (a) Determine a suitable iteration function $T(x)$ such that ξ is a solution of $x = T(x)$ and $T(x)$ is a contraction over $[1, 2]$.

   (b) Find k such that the n^{th} iterate x_n generated by the equation $x_n = T(x_{n-1})$ for $n \geq 1$, satisfies $|x_n - \xi| \leq k^n|x_0 - \xi|$.

5. Let $f(x)$ be the circular quarter arc given by $f(x) = \sqrt{1-x^2}, 0 \leq x \leq 1$. Approximate $f(x)$ by a straight line $p_1(x)$ in the least squares sense using the weight function $\rho(x) = (1-x^2)^{-1/2},\ 0 \leq x \leq 1$. That is, using the inner product

   $$(u, v) = \int_0^1 \frac{u(x)v(x)}{\sqrt{1-x^2}}\,dx.$$

6. Let $f \in C[1, 2]$ and let P^n be the set of polynomials of degree $\leq n$. Define an inner product on $C[1, 2]$ as $(f, g) = \int_1^2 x^2 f(x)g(x)\,dx$ and norm $\|f\| = (f, f)^{1/2}$. Let $\phi_k(x)_{k=0}^{\infty}$ be orthonormal polynomials with respect to this inner product. The least squares approximation $p_n \in P^n$ to $f \in C[1, 2]$ is given by $p_n(x) = \sum_{k=0}^{n}(f, \phi_k)\phi_k(x)$. Prove that $(f - p_n, q_n) = 0$ for any $q_n \in P^n$.

7. Let $Q(f)$ be the $(N+1)$-point Gaussian quadrature rule over the interval $[a,b]$ such that

$$Q(f) = \sum_{i=0}^{N} w_i f(x_i) \approx I(f) = \int_a^b \rho(x) f(x) dx,$$

where $\rho(x)$ is a real, positive weight function on (a,b). Show that if a and b are finite and f is continuous, then $Q(f) \to I(f)$ as $N \to \infty$.

8. Consider the following two-step method,

$$y_{k+1} + \alpha_0 y_k = h\left(\beta_1 f(t_k, y_k) + \beta_0 f(t_{k-1}, y_{k-1})\right),$$

for solving the initial value problem $y'(t) = f(t,y)$.

(a) Find $\alpha_0, \beta_0, \beta_1$ such that the method is second order.

(b) Clearly define consistent, absolutely stable, region of absolute stability, and A-stable.

(c) Is the given two-step method consistent? Why or why not?

9. Consider the nonlinear boundary value problem,

$$-u'' = \cos(u),$$

posed on $(0,1)$ with boundary conditions $u(0) = u(1) = 0$.

(a) Write down the nonlinear algebraic system of equations resulting from the finite difference method with N internal nodes.

(b) Consider the iterative strategy of solving the nonlinear system from (a) with an initial solution vector $u^{(0)}$ and iterating $Au^{(n+1)} = F(u^{(n)})$, where A is the finite difference matrix obtained by discretizing $-u''$ and $F(u^{(n)})_i = \cos u_i^{(n)}$. Show that this iteration converges to the solution of the algebraic equation for any initial input. (Hint: the fact that the eigenvalues of A are known to be $\{2N^2\left(1 - \cos\left(\frac{\pi j}{N+1}\right)\right)\}_{j=1}^{N}$ may be helpful in determining the norm of A^{-1} and/or A.)