Numerical Analysis Preliminary Examination, May 2018

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Solve 8 out of 9 (eight out of nine) of the following problems. Clearly indicate which eight are to be graded.

1. (a) Show that if an $n \times n$ matrix A can be written as $A = LL^T$ where L is a nonsingular, real, lower triangular matrix, then A is real symmetric and positive definite.

   (b) Let $A = \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix}$. Find the Cholesky decomposition of A.

2. Suppose that matrix A is nonsingular, $\boldsymbol{x}$ is the solution of $A\boldsymbol{x} = \boldsymbol{b}$, $\|A^{-1}\|_2 = 10^3$, and $\|A\|_2 = 10^2$. We wish to solve $B\boldsymbol{z} = \boldsymbol{b}$ where $B = A - C$ and $\|C\|_2 = 10^{-4}$.

   (a) Prove that B is nonsingular.

   (b) Find an upper bound on $\|\boldsymbol{x} - \boldsymbol{z}\|_2$ in terms of $\|\boldsymbol{x}\|_2$, that is, find $c > 0$ such that $\|\boldsymbol{x} - \boldsymbol{z}\|_2 < c\|\boldsymbol{x}\|_2$.

3. For the solution of the linear system $Ax = b$ with nonsingular matrix A, consider iterative methods of splitting type

   $$Px^{(k+1)} = Nx^{(k)} + b\,, \quad A = P - N\,.$$

   Assume that P is nonsingular. Then, the iterative scheme may be written as

   $$x^{(k+1)} = Bx^{(k)} + f\,.$$

   Give the expressions of B and f for the Jacobi method and for its version with relaxation parameter.

   Prove that if A is strictly diagonally dominant by rows and P is chosen as in the Jacobi method, then the method is convergent.

4. Let

   $$F(x) = x + f(x)g(x)\,,$$

   where $f(r) = 0$ and $f'(r) \neq 0$. Find the precise conditions on the function g so that the iteration

   $$x_{k+1} = F(x_k)\,, \quad k = 0, 1, 2, \ldots$$

   will converge cubically to r if started near r.

5. Let $f \in C^\infty(\mathbb{R})$ and let $x_0 \in \mathbb{R}$ be given.

   a) Prove that

   $$C_h = \frac{f(x_0 + h) - f(x_0 - h)}{2h} = f'(x_0) + \sum_{i=1}^{\infty} c_i h^{2i}$$

   where $c_i, i = 1, 2, 3, \ldots$ are independent of h.

   b) Suppose that C_h and $C_{h/2}$ have been calculated. Find constants α_1 and α_2 so that

   $$\alpha_1 C_h + \alpha_2 C_{h/2} = f'(x_0) + \mathcal{O}(h^4)\,.$$

6. (a) Consider the formula

   $$\int_0^h f(x)\,dx = h\left\{Af(0) + Bf\left(\frac{h}{3}\right) + Cf(h)\right\}\,.$$

   Find A, B, C such that this is exact for all polynomials of degree less than or equal to 2.

   (b) Suppose that the Trapezoidal rule applied to $\int_0^2 f(x)\,dx$ gives the value $\frac{1}{2}$ while the quadrature rule in part (a) applied to $\int_0^2 f(x)\,dx$ gives the value $\frac{1}{4}$. If $f(0) = 3$, then show that $f\left(\frac{2}{3}\right) = 1$.

7. Given a function $f \in C^0([a,b])$, define the following:

   (a) the Lagrange interpolating polynomial of degree at most n through a set of $n+1$ distinct nodes in $[a,b]$;

   (b) the best approximation to f from the space of polynomials of degree at most n with respect to a norm induced by an inner product.

   Prove both existence and uniqueness of either (a) or (b).

8. *Backward differentiation formulas* (BDF) are linear multistep methods for the numerical solution of

$$y' = f(t,y)\,, \quad y(0) = y_0\,.$$

   These schemes are obtained by considering the Lagrange interpolating polynomial through the discretization points $t_{n+1}, t_n, \ldots, t_{n-s}$ and by evaluating the derivative of this polynomial at t_{n+1}.

   In view of this, derive the 2-step BDF formula

$$y_{n+1} - \frac{4}{3}y_n + \frac{1}{3}y_{n-1} = \frac{2}{3}hf(t_{n+1}, y_{n+1})\,.$$

9. Consider the boundary-value problem

$$\begin{aligned} &-u''(x) + u(x) = f(x) \\ &u(0) = u(1) = 0\,. \end{aligned}$$

   Derive a weak formulation of this problem and identify a function space for the unknown function and the test function for which the integrals in the weak formulation exist.

   Then, derive the linear system arising from a finite dimensional approximation of the above weak problem.