Numerical Analysis Preliminary Examination, August 2018

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Solve 8 out of 9 (eight out of nine) of the following problems. Clearly indicate which eight are to be graded.

1. Let A be a nonsingular $n \times n$ real matrix and $\|A^{-1}B\| = r < 1$.

   (a) Show that $A + B$ is nonsingular and $\|(A+B)^{-1}\| \le \dfrac{\|A^{-1}\|}{1-r}$.

   (b) Show that
   $$\|(A+B)^{-1} - A^{-1}\| \le \frac{\|B\|\,\|A^{-1}\|^2}{1-r}.$$

2. Let $U\boldsymbol{x} = \boldsymbol{b}$ where U is an $n \times n$ nonsingular upper triangular matrix. The vector $\boldsymbol{x}$ can be computed with the algorithm
   $$x_n = b_n/u_{n,n}\,,$$
   $$x_k = (b_k - \sum_{j=k+1}^{n} u_{k,j}x_j)/u_{k,k}\,, \quad k = n-1, n-2, \cdots, 1\,.$$
   Prove that this algorithm requires exactly $(n^2 - n)/2$ subtractions and additions.

3. Let
   $$F(h) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$$
   be an approximation to $f''(x_0)$. Let $e(h) = f''(x_0) - F(h)$ be the error in the approximation. Assume that $f \in C^8([a,b])$ and that $x_0 - h$, x_0, $x_0 + h \in [a,b]$. Prove that the error $e(h)$ has the form
   $$e(h) = c_1h^2 + c_2h^4 + \mathcal{O}(h^6)$$
   where c_1 and c_2 are independent of h.

4. Consider the iteration method
   $$x_{k+1} = \phi(x_k)\,, \quad k = 0, 1, \ldots$$
   for solving the equation $f(x) = 0$. Choose the iteration function of the form
   $$\phi(x) = x - \gamma_1 f(x) - \gamma_2 (f(x))^2$$
   and find γ_1 and γ_2 such that the iteration method is at least of the third order. (Suppose that there is a $\xi \in \mathbb{R}$ such that $f(\xi) = 0$, $f'(\xi) \neq 0$, and $f''(\xi) \neq 0$ with $f \in C^2(\mathbb{R})$.)

5. (a) Find the constants A and B such that the formula
   $$\int_0^{2\pi} f(x)\,dx \approx Af(0) + Bf(\pi)$$
   is exact for any function of the form $f(x) = a + b\cos(x)$.

   (b) Prove that the resulting formula is exact for any function of the form
   $$f(x) = \sum_{k=0}^{n} [a_k \cos((2k+1)x) + b_k \sin(kx)]\,.$$

6. Given a function $f \in C^0([a,b])$, define the following:

   (a) the Hermite interpolating polynomial (i.e., based on the evaluations of the function and its first derivative) of degree at most $2n+1$ through a set of $n+1$ distinct nodes in $[a,b]$;

   (b) the best approximation to f from the space of polynomials of degree at most $2n+1$ with respect to a norm induced by an inner product.

   Prove both existence and uniqueness of either (a) or (b).

7. Consider *Heun's method* for the approximate solution of a scalar initial-value problem

$$y' = f(x, y)\,, \quad y(x_0) = y_0\,, \quad f \in C^{(2,2)}\,,$$

with stepsize h. A step of Heun's method is given by

$$\begin{aligned} \widetilde{y}_n &= y_n + hf(x_n, y_n)\,, \\ x_{n+1} &= x_n + h\,, \\ y_{n+1} &= y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, \widetilde{y}_n)]\,. \end{aligned}$$

(a) State carefully definitions of the following:

i. absolute stability;

ii. A-stability;

iii. local truncation error.

(b) Find the real part of the region of absolute stability for Heun's method. Can you determine from this whether Heun's method is A-stable?

(c) Prove that the local truncation error of Heun's method is $\mathcal{O}(h^3)$.

8. Adams-Moulton methods are implicit linear multistep methods for the numerical solution of

$$y' = f(t, y)\,, \quad y(0) = y_0\,.$$

They are obtained from the integral representation of the solution to this problem, by approximating the integral of the right-hand side using an interpolatory quadrature rule through a given set of nodes.

In view of this, derive the 2-step Adams-Moulton formula

$$y_{n+1} = y_n + \frac{h}{12}\left[5f_{n+1} + 8f_n - f_{n-1}\right].$$

9. Consider the two-point boundary-value problem

$$-y''(x) + q(x)y(x) = r(x),\ 0 < x < 1,\ y(0) = y(1) = 0.$$

Consider the difference scheme

$$\frac{(y_{j+1} - 2y_j + y_{j-1})}{h^2} - q(x_j)y_j = r(x_j),$$

for $j = 1, 2, \cdots, N-1$, with $y_0 = y_N = 0$, $x_j = jh$, and $h = 1/N$.

(a) Determine the $(N-1) \times (N-1)$ matrix A so that the above difference equations can be written as the linear system $A\boldsymbol{y} = h^2\boldsymbol{r}$ with $\boldsymbol{y} = [y_1, y_2, \cdots, y_{N-1}]^T$ and $\boldsymbol{r} = [r(x_1), r(x_2), \cdots, r(x_{N-1})]^T$.

(b) Give a condition on $q(x)$ such that the matrix A is strictly diagonally dominant by rows.