Instructions: Solve any **eight out of nine** of the following problems. All problems have equal weights and the passing score will be determined after all the exams are graded. Indicate clearly which eight problems that you wish to be graded. No textbook. No calculator. No computer. No notes.

1. Let $\|\cdot\|$ be any induced matrix norm. Prove that if E is an $n \times n$ matrix for which $\|E\|$ is sufficiently small, then
$$\left\|(I-E)^{-1}-(I+E)\right\| \leq 3\|E\|^2.$$
Determine how small $\|E\|$ should be.

2. Let $x \in \mathbb{R}^n$ with $x = [x_1, x_2, x_3, \ldots, x_n]^T$ and $x_1 \neq 0$.
Let
$$\sigma = \operatorname{sign}(x_1)\|x\|_2, \quad u = x + \sigma e_1, \quad \theta = \frac{1}{2}\|u\|_2^2, \quad U = I - uu^T/\theta.$$
Prove that
$$U^2 = I, \quad \text{and} \quad Ux = -\sigma e_1,$$
where $e_1 = [1, 0, 0, \ldots, 0]^T$.

3. Let A be an $m \times n$ matrix, $m \geq n$, having full rank.
   (a) Find x that minimizes $\|Ax - b\|_2$.
   (b) Show that
$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \cdot \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$
   has a solution where x is the minimizer in (a).

4. Consider the fixed-point iteration method
$$x_{k+1} = \phi(x_k), \quad k = 0, 1, \ldots$$
for solving the equation $f(x) = 0$. Choose the auxiliary iteration function of the form
$$\phi(x) = x - \gamma_1 f(x) - \gamma_2 (f(x))^2.$$
Find γ_1 and γ_2 such that the iteration method is at least of the third order. (Suppose that there is a $\xi \in \mathbb{R}$ such that $f(\xi) = 0$, $f'(\xi) \neq 0$, $f''(\xi) \neq 0$ with $f \in C^2(\mathbb{R})$).

5. Given $n+1$ distinct points $\{x_0, x_1, \ldots, x_n\}$, let
$$l_j(x) = \prod_{i=0, i\neq j}^{n} \frac{x - x_i}{x_j - x_i}, \quad j = 0, 1, \cdots n$$
be the Lagrange characteristic functions. Show that
$$\sum_{j=0}^{n} x_j^k l_j(x) = x^k, \quad k = 0, 1, \ldots, n.$$

6. Let $\mathbb{P}^N$ be the space of polynomials of degree less than or equal to N. Given a function $f \in C^0([-1,1])$, let p_N be the best approximation of f in $\mathbb{P}^N$ with respect to the L^2 norm:

$$p_N = \arg\min_{g \in \mathbb{P}^N} \|f - g\|_{L^2([-1,1])}.$$

   (a) Show that p_N satisfies

$$(p_N, g) = (f, g), \quad \forall g \in \mathbb{P}^N,$$

   where the inner product $(\cdot, \cdot)$ is defined by

$$(u, v) = \int_{-1}^{1} u(x)v(x)dx.$$

   (b) Given $f(x) = x^3 + 1$ and $N = 1$, solve the best approximation $p_1(x)$.

7. The Hermite polynomials obey the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

   and are orthogonal with respect to the inner product

$$(u, v) = \int_{-\infty}^{\infty} e^{-x^2} u(x)v(x)dx$$

   (a) Find the nodes and weights for the two-point Gaussian quadrature rule for the approximation of

$$I(f) = \int_{-\infty}^{\infty} e^{-x^2} f(x)dx.$$

   You may need the formula $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

   (b) For what space of functions f is this two-point rule exact?

8. Consider the Crank-Nicolson method (CN) for the approximate solution of a scalar initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0, \quad f \in C^{(2,2)},$$

   with stepsize h. A step of the CN method is given by

$$\begin{aligned} t_{n+1} &= t_n + h \\ u_{n+1} &= u_n + \frac{h}{2}\left[f(x_n, u_n) + f(t_{n+1}, u_{n+1})\right]. \end{aligned}$$

   (a) Prove that the local truncation error of the CN method is $O(h^2)$.

   (b) Find the region of absolute stability for the CN method. Is the CN method A-stable? Why?

9. Consider a multistep method of the form

$$u_{n+1} = a_1 u_n + a_2 u_{n-1} + h\left(b_0 f\left(t_{n+1}, u_{n+1}\right) + b_1 f\left(t_n, u_n\right)\right)$$

for solving the ODE $y' = f(t, y)$.

(a) Determine the coefficients a_1, a_2, b_0, b_1 that make the method reach the highest order of accuracy it can attain.

(b) Determine whether or not the obtained method satisfies the root condition.