Instructions: Solve any **eight out of nine** of the following problems. All problems have equal weights and the passing score will be determined after all the exams are graded. Indicate clearly which eight problems that you wish to be graded. No textbook. No calculator. No computer. No notes.

1. A and B are $n \times n$ matrices. Show that if $\|AB - I\| = \epsilon < 1$, then A and B are both nonsingular, and
$$\|A^{-1} - B\| \leq \|B\| \left(\frac{\epsilon}{1-\epsilon} \right).$$

2. (a) Let A be an $n \times n$ matrix. Show that $\|A^T A\|_2 = \|A\|_2^2$ and $\kappa_2\left(A^T A\right) = \kappa_2(A)^2$, where $\kappa_2(A)$ denotes the 2-norm condition number of matrix A.
   (b) Let M be an $n \times n$ positive definite matrix, and L is its Cholesky factor so that $M = LL^T$. Show that $\|M\|_2 = \|L\|_2^2$ and $\kappa_2(M) = \kappa_2(L)^2$.

3. A is a $(p+q) \times (p+q)$ nonsingular matrix given by
$$A = \begin{bmatrix} C & B^T \\ B & 0 \end{bmatrix},$$
where C is a nonsingular upper-triangular $p \times p$ matrix, and B is a $q \times p$ matrix. Assume A has an LU decomposition with
$$L = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}, \quad U = \begin{bmatrix} C & Y \\ 0 & Z \end{bmatrix}.$$
Find X, Y, Z.

4. Consider the fixed-point iterative method
$$x_{n+1} = F(x_n) = x_n + f(x_n)/g(x_n), \quad n = 0, 1, \ldots.$$
Assume that the method converges to a point α which is a simple root of the function $f(x)$ but not a root of the function $g(x)$. Find $g(\alpha)$ and $g'(\alpha)$ in terms of $f(\alpha)$, $f'(\alpha)$, and $f''(\alpha)$ so that the method is at least of the third order.

5. Given $n+1$ distinct data points
$$\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))\}.$$
Let
$$l_j(x) = \prod_{i=0, i\neq j}^{n} \frac{x - x_i}{x_j - x_i}, \quad j = 0, 1, \cdots n$$
be the Lagrange characteristic functions. The Lagrange form of polynomial interpolation is given by
$$p_n(x) = \sum_{j=0}^{n} f(x_j) l_j(x).$$
Show that

(a) $\sum_{j=0}^{n} l_j(x) = 1$.

(b)

$$p_n(x) = \omega_{n+1}(x) \sum_{j=0}^{n} \frac{w_j}{x - x_j} f(x_j),$$

where

$$\omega_{n+1}(x) = \prod_{j=0}^{n}(x - x_j) \quad \text{and} \quad w_j = \prod_{i=0, i\neq j}^{n} \frac{1}{(x_j - x_i)}.$$

(c)

$$p_n(x) = \frac{\displaystyle\sum_{j=0}^{n} \frac{w_j}{x - x_j} f(x_j)}{\displaystyle\sum_{j=0}^{n} \frac{w_j}{x - x_j}}.$$

6. Let $\mathbb{P}^N$ be the space of polynomials of degree less than or equal to N. Given a function $f \in C^0([0,1])$, let p_N be the best approximation of f in $\mathbb{P}^N$ with respect to the L^2 norm:

$$p_N = \arg\min_{g \in \mathbb{P}^N} \|f - g\|_{L^2([0,1])}.$$

(a) Show that p_N satisfies

$$(p_N, g) = (f, g), \quad \forall g \in \mathbb{P}^N,$$

where the inner product $(\cdot, \cdot)$ is defined by

$$(u, v) = \int_0^1 u(x)v(x)dx.$$

(b) Given $f(x) = x^3$ and $N = 1$, solve the best approximation $p_1(x)$.

7. Consider a quadrature formula of the type

$$I(f) = \int_0^\infty e^{-x} f(x)\, dx \approx af(0) + bf(c).$$

(a) Find a, b and c such that the formula is exact for polynomials of the highest degree possible. (Note that $\int_0^\infty e^{-x}\, x^n\, dx = n!$.)

(b) Let $P(x) \in \mathbb{P}^2$ be the Hermite polynomial interpolating f with constraints $P(0) = f(0)$, $P(2) = f(2)$ and $P'(2) = f'(2)$. Determine $\int_0^\infty e^{-x}\, P(x)\, dx$ and compare with the result in part (a).

8. Consider the Heun's method for the approximate solution of a scalar initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0, \quad f \in C^{(2,2)},$$

with stepsize h. A step of the Heun's method is given by

$$\begin{aligned} t_{n+1} &= t_n + h \\ u_{n+1} &= u_n + \frac{h}{2}\left[f(x_n, u_n) + f(t_{n+1}, u_n + hf(t_n, u_n))\right]. \end{aligned}$$

   (a) Prove that the local truncation error of the Heun's method is $O(h^2)$.

   (b) Find the region of absolute stability for the Heun's method. Is the Heun's method A-stable? Why?

9. Consider a multistep method of the form

$$u_{n+1} = a_1 u_n + a_2 u_{n-1} + h\left(b_0 f(t_{n+1}, u_{n+1}) + b_1 f(t_n, u_n) + b_2 f(t_{n-1}, u_{n-1})\right)$$

for solving the ODE $y' = f(t, y)$.

   (a) Determine the coefficients a_1, a_2, b_0, b_1, b_2 that make the method reach the highest order of accuracy it can attain.

   (b) Determine whether or not the obtained method satisfies the root condition.