DO ALL NINE PROBLEMS

Problem 1. Consider the \( n \times n \) tridiagonal matrix

\[
A_n = (n+1)^2 \begin{bmatrix}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
& -1 & 2 & -1 & \\
& & \ddots & \ddots & \\
& & & -1 & 2 \\

dotfill
\end{bmatrix}
\]

You may use without proof the fact that the eigenvalues of this matrix are

\[
\lambda_j = 2(n+1)^2 \left[ 1 - \cos \left( \frac{j\pi}{n+1} \right) \right] ; \quad j = 1, 2, \ldots, n.
\]

1. Compute \( \|A_n\|_1 \), \( \|A_n\|_\infty \), and \( \|A_n\|_2 \).
2. In one of the three norms \( \|\cdot\|_1 \), \( \|\cdot\|_\infty \), and \( \|\cdot\|_2 \), it is convenient to compute both \( \|A_n\| \) and \( \|A_n^{-1}\| \). Using that norm, do the following:
   (a) Prove that \( \|A_n^{-1}\| < 1 \) for all \( n = 1, 2, 3, \ldots \).
   (b) Prove that \( \|I - \frac{1}{2(n+1)^2}A_n\| < 1 \) for all \( n = 1, 2, 3, \ldots \).

Problem 2. Use any method you like to find the least squares solution to \( Ax = b \), where

\[
A = \begin{bmatrix}
0 & 1 \\
1 & 1 \\
0 & 2
\end{bmatrix} ; \quad b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.
\]

Problem 3. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( f(x) = (1 + e^x)^{-1} \).

1. Compute the condition number \( \kappa_f(x) \) for evaluation of this function.
2. Assuming that one somehow has an exact algorithm for computing \( f \) at any input, and assuming that the exact real input \( x \) is represented approximately by a standard double precision floating point (DPFP) number \( \tilde{x} \), estimate the error \( |f(x) - f(\tilde{x})| \).
3. Identify where (if anywhere) accurate computation of \( f \) will be effectively impossible in DPFP; for example, regions such as \( x \ll -1, x \approx 0 \) and so on. Assume you can ignore considerations of overflow and underflow.

Problem 4. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( f(x) = (x + 4)^{-1} \). Prove that iteration \( x_{n+1} = f(x_n) \) converges to a fixed point of \( f \) starting from any \( x_0 \geq 0 \), and find that fixed point.

Problem 5. Let \( A \) be an \( m \times n \) complex matrix.

1. What, if any, conditions must be placed on \( A \) for \( A \) to have a full singular value decomposition (SVD) \( A = U \Sigma V^* \)? (The notation \( V^* \) indicates the conjugate transpose of \( V \).)
2. What are the properties (sizes and shapes, any special structures, etc) of the factors \( U, \Sigma, \) and \( V \)?
3. Assuming \( A \) has a full SVD, outline a method for computing the factors \( U, \Sigma, \) and \( V \). (There are several such methods; choose any one you like).
Problem 6. Let $f$ be the cubic polynomial $f(x) = 1 + x^3$. The space $\mathbb{P}^n$ is the set of polynomials of degree $\leq n$.

1. Find the degree two Lagrange interpolant to $f$ based on the nodes $x_0 = -1, x_1 = 0$, and $x_2 = 1$. 
2. Find the best approximation to $f$ from $\mathbb{P}^2$ in the norm
\[ \| v \| = \sqrt{\int_{-1}^{1} (v(x))^2 \, dx}. \]

Helpful fact: $\int_{-1}^{1} P_n(x)^2 \, dx = \frac{2}{2n+1}$, where $P_n(x)$ is the degree $n$ Legendre polynomial.

**Problem 7.** Consider a two-point quadrature rule
\[ Q_2(f) = w_1 f(x_1) + w_2 f(x_2) \]
for numerical approximation of the definite integral
\[ I(f) = \int_0^1 \sqrt{x} f(x) \, dx. \]
Assume throughout that the two nodes $x_1$ and $x_2$ are distinct and both are within the interval $[0, 1]$. The space $\mathbb{P}^n$ is the set of polynomials of degree $\leq n$.

1. Prove that $Q_2$ is exact for all $f \in \mathbb{P}^1$ for every choice of nodes satisfying the assumptions above.
2. Find the nodes $x_1, x_2$ and weights $w_1, w_2$ such that $Q_2$ is exact for all $f \in \mathbb{P}^3$.
3. Given differentiability requirements on $f$ (to be stated by you), derive an upper bound on the error $E_2(f) = |I(f) - Q_2(f)|$ for the quadrature method derived in part 2.

**Problem 8.** Consider the 2-stage explicit Runge-Kutta (ERK) method for the IVP $y' = f(t,y), y(t_0) = y_0$. We compute the stage variables $K_1$ and $K_2$ as
\[ K_1 = f(t_n,y_n) \]
\[ K_2 = f(t_n + c_2 h, y_n + h A_{21} K_1), \]
and then advance the approximate solution to time $t_{n+1}$ with the step formula
\[ y_{n+1} = y_n + h (b_1 K_1 + b_2 K_2). \]
Recall that the Butcher coefficients $c_2, A_{21}, b_1, b_2$ are to be determined by a set of equations called the order conditions.

1. State the definitions of local and global truncation error for a Runge-Kutta method.
2. Derive order conditions that must be satisfied for the 2-stage ERK method above to have second order global truncation error.
3. Find a solution to the order conditions, that is, find a choice of coefficients $c_2, A_{21}, b_1, b_2$ that define a specific second order method. This solution is not unique; however, you only need to find one solution.
4. Now consider application of the second order method from part 3 to the initial value problem $y' = -10y, y(0) = 1$. The exact solution $y(t) = e^{-10t}$ decays with time. Find the interval of timesteps $h$ within which the numerical solution with timestep $h$ is decaying.

**Problem 9.** Find the values of $\beta$ for which the matrix
\[ A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 2 & 4 & \beta \end{bmatrix} \]
is positive definite.