Numerical analysis preliminary exam
August 2020

DO ALL NINE PROBLEMS

Problem 1. Consider the $n \times n$ tridiagonal matrix

$$A_n = (n+1)^2 \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

You may use without proof the fact that the eigenvalues of this matrix are

$$\lambda_j = 2\,(n+1)^2 \left[1 - \cos\left(\frac{j\pi}{n+1}\right)\right]; \quad j = 1, 2, \cdots, n.$$

1. Compute $\|A_n\|_1$, $\|A_n\|_\infty$, and $\|A_n\|_2$.
2. In one of the three norms $\|\cdot\|_1$, $\|\cdot\|_\infty$, and $\|\cdot\|_2$, it is convenient to compute both $\|A_n\|$ and $\left\|A_n^{-1}\right\|$. Using that norm, do the following:
   (a) Prove that $\left\|A_n^{-1}\right\| < 1$ for all $n = 1, 2, 3, \cdots$.
   (b) Prove that $\left\|I - \frac{1}{2(n+1)^2} A_n\right\| < 1$ for all $n = 1, 2, 3, \cdots$.

Problem 2. Use any method you like to find the least squares solution to $Ax = b$, where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}; \quad b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = (1 + e^x)^{-1}$.

1. Compute the condition number $\kappa_f(x)$ for evaluation of this function.
2. Assuming that one somehow has an *exact* algorithm for computing f at any input, and assuming that the exact real input x is represented approximately by a standard double precision floating point (DPFP) number $\tilde{x}$, estimate the error $|f(x) - f(\tilde{x})|$.
3. Identify where (if anywhere) accurate computation of f will be effectively impossible in DPFP; for example, regions such as $x \ll -1$, $x \approx 0$ and so on. Assume you can ignore considerations of overflow and underflow.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = (x+4)^{-1}$. Prove that iteration $x_{n+1} = f(x_n)$ converges to a fixed point of f starting from any $x_0 \geq 0$, and find that fixed point.

Problem 5. Let A be an $m \times n$ complex matrix.

1. What, if any, conditions must be placed on A for A to have a full singular value decomposition (SVD) $A = U\Sigma V^*$? (The notation V^* indicates the conjugate transpose of V.)
2. What are the properties (sizes and shapes, any special structures, etc) of the factors U, Σ, and V?
3. Assuming A has a full SVD, outline a method for computing the factors U, Σ, and V. (There are several such methods; choose any one you like).

Problem 6. Let f be the cubic polynomial $f(x) = 1 + x^3$. The space $\mathbb{P}^n$ is the set of polynomials of degree $\leq n$.

1. Find the degree two Lagrange interpolant to f based on the nodes $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.

2. Find the best approximation to f from $\mathbb{P}^2$ in the norm

$$\|v\| = \sqrt{\int_{-1}^{1} (v(x))^2 \, dx}.$$

Helpful fact: $\int_{-1}^{1} P_n(x)^2 \, dx = \frac{2}{2n+1}$, where $P_n(x)$ is the degree n Legendre polynomial.

Problem 7. Consider a two-point quadrature rule

$$Q_2(f) = w_1 f(x_1) + w_2 f(x_2)$$

for numerical approximation of the definite integral

$$I(f) = \int_0^1 \sqrt{x} \, f(x) \, dx.$$

Assume throughout that the two nodes x_1 and x_2 are distinct and both are within the interval $[0,1]$. The space $\mathbb{P}^n$ is the set of polynomials of degree $\leq n$.

1. Prove that Q_2 is exact for all $f \in \mathbb{P}^1$ for every choice of nodes satisfying the assumptions above.
2. Find the nodes x_1, x_2 and weights w_1, w_2 such that Q_2 is exact for all $f \in \mathbb{P}^3$.
3. Given differentiability requirements on f (to be stated by you), derive an upper bound on the error $E_2(f) = |I(f) - Q_2(f)|$ for the quadrature method derived in part 2.

Problem 8. Consider the 2-stage explicit Runge-Kutta (ERK) method for the IVP $y' = f(t,y)$, $y(t_0) = y_0$. We compute the stage variables K_1 and K_2 as

$$K_1 = f(t_n, y_n)$$
$$K_2 = f(t_n + c_2 h, y_n + hA_{21}K_1),$$

and then advance the approximate solution to time t_{n+1} with the step formula

$$y_{n+1} = y_n + h(b_1 K_1 + b_2 K_2).$$

Recall that the Butcher coefficients c_2, A_{21}, b_1, and b_2 are to be determined by a set of equations called the order conditions.

1. State the definitions of local and global truncation error for a Runge-Kutta method.
2. Derive order conditions that must be satisfied for the 2-stage ERK method above to have second order global truncation error.
3. Find a solution to the order conditions, that is, find a choice of coefficients c_2, A_{21}, b_1, and b_2 that define a specific second order method. This solution is not unique; however, you only need to find *one* solution.
4. Now consider application of the second order method from part 3 to the initial value problem $y' = -10y$, $y(0) = 1$. The exact solution $y(t) = e^{-10t}$ decays with time. Find the interval of timesteps h within which the numerical solution with timestep h is decaying.

Problem 9. Find the values of β for which the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 2 & 4 & \beta \end{bmatrix}$$

is positive definite.