Numerical analysis preliminary exam
January 2021

DO ALL TEN PROBLEMS

Problem 1. Consider an inner product $(\cdot,\cdot)$ and norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$ defined on the set of real-valued functions of one variable. Let V be the space of functions for which this norm is finite; you may assume this space is complete. Let $\{\psi_n(x)\}_{n=1}^{\infty}$ be a set of functions that are orthonormal in this inner product.

1. Find the coefficients c_n such that

$$f_N(x) = \sum_{n=1}^{N} c_n \psi_n(x)$$

   is the best approximation to $f \in V$ in the norm $\|\cdot\|$.

2. Suppose for some function $f \in V$ the expansion coefficients are $c_n = e^{-n}$. Derive a bound on the error $\|f - f_N\|$.

Problem 2. Let $\|\cdot\|$ denote both a vector norm on $\mathbb{C}^m$ and the induced matrix norm on $\mathbb{C}^{m\times m}$. Let $\rho(A)$ be the spectral radius of A, defined by

$$\rho(A) = \max_{i=1,m} |\lambda_i|$$

where λ_i is the i-th eigenvalue of A.

1. Prove the inequality $\|A\| \geq \rho(A)$.

2. Prove that if $\|I - A\| < 1$, then A is nonsingular.

Problem 3. Use any method you like to find a full QR factorization of the matrix

$$A = \begin{bmatrix} 0 & 4 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}.$$

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos(x)$.

1. Compute the condition number $\kappa_f(x)$ for evaluation of this function.

2. Show that $\kappa_f(x) \gg 1$ when $x \approx \frac{\pi}{2}$. Although the condition number is large for $x \approx \frac{\pi}{2}$, the cosine can be computed near $\frac{\pi}{2}$ in floating point to high accuracy (small absolute error). Explain.

3. Is it possible to compute $\cos(x)$ accurately in double precision floating point for $x \gtrsim 10^{16}$? Explain.

Problem 5. Find the degree 2 Lagrange interpolant $p_2(x)$ to $f(x) = \cos\left(\frac{\pi x}{2}\right)$ based on the points $\{-1, 0, 1\}$, then find a bound on the error norm $\|f(x) - p_2(x)\|_{\infty}$.

Problem 6. Starting with the initial guess $x_0 = \frac{7}{5}$, estimate $\sqrt{2}$ by carrying out one iteration of Newton's method applied to the equation $x^2 - 2 = 0$.

Problem 7. In the computation of an LU factorization of an $m \times m$ matrix by Gaussian elimination, it is necessary to evaluate very efficiently the product

$$L_1^{-1} L_2^{-1} \cdots L_{m-1}^{-1} L_m^{-1}$$

where each L_k is an $m \times m$ lower triangular matrix with unit diagonal and zeros below the diagonal except in column k. For example, the 5×5 matrix L_2 is

$$L_2 = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & \ell_{32} & 1 & & \\ & \ell_{42} & & 1 & \\ & \ell_{52} & & & 1 \end{bmatrix}.$$

1. For general m and k, write L_k as the identity plus a sum of $m-k$ rank one matrices.
2. For general $k < m$, write L_k^{-1} as the identity plus a sum of $m-k$ rank one matrices. Verify that $L_k L_k^{-1} = I$ and show that no operations are needed to compute L_k^{-1}.
3. For general $k < m$, show how to compute $L_k^{-1} L_{k+1}^{-1}$ without doing any operations.

Problem 8. Find a full singular value decomposition of the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}.$$

Problem 9. Let $\rho(x)$ be positive on (a, b). Find the node x_1 and weight w_1 so that the one point quadrature rule

$$Q_1(f) = w_1 f(x_1)$$

computes

$$I(f) = \int_a^b \rho(x) f(x)\, dx$$

exactly for all $f \in \mathbb{P}^1$.

Problem 10. Consider the 1-stage implicit Runge-Kutta (IRK) method for the IVP $y' = f(t, y)$, $y(t_0) = y_0$. We compute the stage variable K_1 by solving the equation

$$K_1 = f(t_n + c_1 h,\, y_n + h A_{11} K_1),$$

and then advance the approximate solution to time t_{n+1} with the step formula

$$y_{n+1} = y_n + h b_1 K_1.$$

Recall that the Butcher coefficients c_1, A_{11}, and b_1, are to be determined by a set of equations called the order conditions.

1. State the definitions of local and global truncation error for a Runge-Kutta method.
2. Derive order conditions that must be satisfied for the 1-stage IRK method above to have third order *local* truncation error.
3. The implicit midpoint method has $c_1 = \frac{1}{2}$, $A_{11} = \frac{1}{2}$, and $b_1 = 1$. Prove this method satisfies the order conditions from part 2.
4. Is the implicit midpoint method A-stable?