Numerical Analysis Preliminary Exam

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Instructions: Solve **all nine** of the following problems.

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1. (a) Let $A \in \mathbb{R}^{m\times n}$ have a singular value decomposition $A = U\Sigma V^T$. Let α be any positive real number. Find the singular value decomposition of $\alpha I + A^T A$.

   (b) Let $A = \begin{bmatrix} 3 & 0 \\ -4 & 0 \end{bmatrix}$.

      i. Compute a singular value decomposition of A.

      ii. Determine $\|A\|_1$, $\|A\|_2$,$\|A\|_\infty$, and $\|A\|_F$.

2. Let $A \in \mathbb{R}^{m\times m}$ be symmetric positive definite. Consider the function $f(x) = \frac{1}{2}x^T Ax - x^T b$.

   (a) Prove that x is the minimizer of f if and only if $Ax = b$.

   (b) One of the simplest iterative procedures to minimize f, and thereby solve $Ax = b$, is Cauchy's method of Steepest Descent. Given a current solution estimate x_k:

      i. Find the direction of steepest descent, p_k.

      ii. Find the line search parameter α_k such that $x_{k+1} = x_k + \alpha_n p_k$ is the minimizer along the direction of steepest descent.

3. Let $A \in \mathbb{R}^{m\times m}$ and suppose $A = QR$ is a QR factorization of A.

   (a) Show that the cost of computing $A = QR$ via Gram-Schmidt is asymptotically $2m^3$ flops (floating point operations).

   (b) Show that the matrix RQ is similar to A.

4. A fast method for evaluating the reciprocal of $a \neq 0$ is to apply a variant of Newton's method to solve the equation $ax - 1 = 0$. The Newton step formula is

$$x_{n+1} = x_n - a^{-1}(ax_n - 1) = a^{-1},$$

   which as written is useless since it requires a^{-1}, exactly what we're trying to compute! The trick is to approximate a^{-1} in the step formula by x_n, yielding the modified Newton step

$$x_{n+1} = x_n(2 - ax_n).$$

   Note that no divisions are needed in this formula.

   (a) Carry out two modified Newton steps to estimate 3^{-1} beginning with the initial estimate $x_0 = \frac{3}{10}$. Confirm that convergence appears quadratic.

   (b) Prove that this method converges quadratically to a^{-1} starting from any x_0 for which $|ax_0 - 1| < 1$.

5. Let $\|\cdot\|$ be the norm induced by the inner product

$$(u, v) = \int_{-1}^{1} u(x)\, v(x)\, dx.$$

   Find the best cubic approximation to $f(x) = x^5$ in this norm.

6. Consider the following interpolation problem: given a function $f \in C^3$ and two distinct points $x_0 < x_1$, find $p \in P^2$ such that

$$p(x_0) = f(x_0)$$
$$p'(x_0) = f'(x_0)$$
$$p(x_1) = f(x_1).$$

   (a) Prove that this interpolation problem has a unique solution.

   (b) Define the interpolation error $e(x)$ as $f(x) - p(x)$. Prove that for every $x \in [x_0, x_1]$ $\exists \xi \in (x_0, x_1)$ such that the interpolation error is

$$e(x) = \frac{f^{(3)}(\xi)}{6} w(x),$$

   where $w(x) = (x - x_0)^2 (x - x_1)$. Hint: the function $g(x,t) = e(t) - e(x) w(t) / w(x)$ might be useful.

7. Let $Q_3(f)$ be the two point Gauss-Legendre quadrature rule for approximation of

$$I(f) = \int_{-1}^{1} f(x)\, dx.$$

   (a) Find the nodes and weights for Q_3.

   (b) Suppose that $f(x) = \frac{1}{1+x^2}$.

   (c) Compute $Q_3(f)$.

   (d) Derive an upper bound on $|I(f) - Q_3(f)|$.

8. Let $f \in C^\infty[a,b]$ and define

$$f''_{FD} = \frac{f(x_0 + 2h) - 2f(x_0 + h) + f(x_0)}{h^2}.$$

   Prove that in exact arithmetic $f''(x_0) = f''_{FD} + \mathcal{O}(h^2)$ if $x_0,\ x_0 + h,\ x_0 + 2h \in [a,b]$.

9. The implicit midpoint method for the numerical solution of the initial value problem $y' = f(x,y)$, $y(x_0) = y_0$, is a one-stage implicit Runge-Kutta method with a step defined by

$$K_1 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} K_1\right)$$
$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_n + hK_1.$$

   (a) Prove that this method has local truncation error $O(h^3)$.

   (b) Find the method's domain of absolute stability and determine whether this method is A-stable.