Numerical Analysis Preliminary Exam

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Instructions: Solve **all nine** of the following problems.

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1. Let $A \in \mathbb{R}^{m\times m}$ be symmetric positive definite. The Rayleigh quotient is the function

$$R_A(x) = \frac{x^T A x}{x^T x},$$

defined for all $x \neq 0$.

(a) Prove that if v is a stationary point of R_A, then (1) v is an eigenvector of A and (2) the value $R_A(v)$ is the eigenvector corresponding to A.

(b) The smallest eigenvalue and a corresponding eigenvector of

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

are $\lambda_1 = 1, v_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Use this information together with a Rayleigh quotient to estimate the smallest eigenvalue of

$$A_\epsilon = \begin{bmatrix} 2 & -1 \\ -1 & 2+\epsilon \end{bmatrix}.$$

2. (a) Let $A \in \mathbb{R}^{m\times m}$. Prove that A can be written as $A = LL^T$,where $L \in \mathbb{R}^{m\times m}$ is nonsingular and lower triangular, if and only if A is symmetric and positive definite.

(b) Let

$$A = \begin{bmatrix} 9 & -3 & 3 \\ -3 & 10 & 2 \\ 3 & 2 & 6 \end{bmatrix}.$$

Compute the Cholesky factorization of A.

3. Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ and let $A = U\Sigma V^*$ be a full singular value decomposition (SVD) of A.

(a) Describe the sizes and properties of the matrices U, Σ, and V.

(b) Assume that A has full column rank. Find an SVD for $A^+ = (A^*A)^{-1} A^*$ in terms of U, Σ, and V. Explain why it is necessary for A to have full column rank.

(c) If A has full column rank, use the SVD of A to compute the 2-norm condition number of A^*A.

4. Let r be the real cube root of $a > 0$. Note that r is the real root of the function $f(x) = x^3 - a$.

   (a) Apply Newton's method to derive an iterative formula for computing r.

   (b) Starting with $x_0 = 1$ and setting $a = 2$, carry out one iteration to compute an improved estimate x_1 for $2^{1/3}$.

   (c) Find an open interval $G \ni r$ such that Newton's method will converge quadratically to r starting from any $x_0 \in G$.

5. Let $\|\cdot\|$ be the L^2 norm induced by the real inner product

$$\langle u, v \rangle = \int_0^1 u(x)\, v(x)\, dx.$$

   Let P^n be the set of all polynomials of degree less than or equal to n, and suppose that $p_n \in P^n$ is the best L^2 approximation to $f \in C^1[0,1]$ from P^n.

   (a) Prove $\exists\, n+1$ points $x_i \in (0,1)$ such that $p_n(x_i) = f(x_i)$.

   (b) Discuss the application of the result of part 1 to estimating the error $\|f - p_n\|$.

6. Consider first-degree Lagrange interpolation to $f(x) = \log(x)$ based on the points $x_0 = 1$ and $x_1 = 2$.

   (a) Find the Lagrange basis functions $\ell_0(x)$ and $\ell_1(x)$.

   (b) Construct the Lagrange interpolating polynomial $p_1(x)$

   (c) State (without proof) a theorem giving the error $f(x) - p_1(x)$.

   (d) Use the error formula to find an upper bound on $\sup_{x \in [1,2]} |f(x) - p_1(x)|$.

7. Let $f(x) = \left(1 + x^2\right)^{-1}$. Choose a two-point quadrature method Q_2 for the approximation of

$$I(f) = \int_{-1}^{1} f(x)\, dx.$$

   (a) Explain your choice of method.

   (b) Use your chosen method to estimate $I(f)$.

   (c) Find a bound on $|I(f) - Q_2(f)|$.

8. Let $f \in C^4[a,b]$ and define

$$f''_{FD} = \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}.$$

   Prove that in exact arithmetic $f''(x_0) = f''_{FD} + \mathcal{O}\left(h^2\right)$ if $x_0 - h,\ x_0,\ x_0 + h \in [a,b]$.

9. Heun's method for the numerical solution of the initial value problem $y' = f(x,y), y(x_0) = y_0$, is a two-stage explicit Runge-Kutta method with a step defined by

$$\begin{aligned} K_1 &= f(x_n, y_n) \\ K_2 &= f(x_n + h, y_n + hK_1) \\ x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + \frac{h}{2}\left[K_1 + K_2\right]. \end{aligned}$$

   (a) Prove that this method has local truncation error $O\left(h^3\right)$. Clearly state the necessary differentiability conditions on $f(x,y)$.

   (b) Find the real part of the domain of absolute stability for Heun's method. Can you determine from this whether this method is A-stable?