Numerical Analysis Preliminary Examination
August 2023

Complete all of the questions on the exam. At the top of each page, write your assigned number, the problem number and the overall page number.

Problem 1 Consider the problem of solving $A\mathbf{x} = \mathbf{b}$ where A is the matrix

$$A = \begin{bmatrix} 5 & -1 & \frac{3}{2} \\ 0 & -1 & \frac{1}{3} \\ 3 & 3 & 7 \end{bmatrix}.$$

prove that the Jacobi method applied to solve this problem is convergent.

Problem 2 Determine the function $p(t) \in \text{Span}\left(\{cos(\pi t), t\}\right)$ minimizing the least square error

$$\sum_{i=1}^{4}(p(x_i) - y_i))^2 \leq \min_{q(x)\in\mathbb{P}_2} \sum_{i=1}^{4}(q(x_i) - y_i)^2$$

for the (x, y) data $\{(1, -1.9), (3, -1.7), (6, 2.6), (8, 2.8), (11, -0.9), (18, 3.8)\}$

Problem 3 Solve the following problem using the LU decomposition

$$\begin{bmatrix} 2 & 4 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

Problem 4 Compute a reduced SVD of the matrix

$$\begin{bmatrix} 0 & \frac{1}{15} \\ 1 & 0 \\ 0 & \frac{2}{15} \\ 0 & \frac{2}{15} \end{bmatrix},$$

you may find the prime factorization $225 = 5^2 \times 3^2$ useful.

Problem 5 Consider the variable coefficient elliptic boundary value problem with homogeneous Dirichlet boundary conditions on $[0, 1]$ given by

$$-\frac{\partial}{\partial_x}\left((2x+1)\frac{\partial}{\partial_x}u(x)\right) = x \tag{1a}$$

$$u(0) = u(1) = 0. \tag{1b}$$

The second order central differences for approximating the first and second derivatives are given by

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \mathcal{O}(h^2), \qquad u_i' = \frac{u_{i+1} - u_{i-1}}{2h} + \mathcal{O}(h^2).$$

Determine the matrix system $A\mathbf{U} = \mathbf{F}$ resulting from discretizing the system (1a)-(1b) using the second order finite differences, above, on the mesh $\mathcal{T}_h = \{0, 1/4, 1/2, 3/4, 1\}$. You may find it useful to use the product rule $(gu')' = gu'' + g'u'$.

Problem 6 This problem has two parts. The system of nonlinear equations

$$F(\mathbf{x}) = 0, \qquad \text{where } F(\mathbf{x}) = F(x_1, x_2) = \begin{cases} x_1^2 + x_2^2 - 9 \\ x_2 + \frac{1}{2}x_1 - 3/2 \end{cases},$$

has two solutions. Find one solution, α, of the system above.

a) Find a fixed point mapping $G(\mathbf{x})$ for the system, above, and prove that there exists some neighborhood U of α such that the fixed-point iteration

$$\mathbf{x}^{(k+1)} = G(\mathbf{x}^{(k)}),$$

converges to α if the initial iterate is chosen in U.

b) Apply one step of Newton's method, computing $\mathbf{x}^{(1)}$, using the initial iterate $\mathbf{x}^{(0)} = [\frac{5}{2}, 0]^T$.

Problem 7 State the equations for the explicit Runge-Kutta method

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 5/6 & 5/6 & 0 \\ \hline & 2/5 & 3/5 \end{array}$$

for solving the initial value problem: $y'(t) = f(t, y(t))$, where $y(0) = y_0$ and $0 \le t \le 1$, and show this method is of order two.

Problem 8 This problem has two parts.

a) Let f be a continuous function defined on the interval $[a, b]$. Derive the quadrature rule corresponding to approximating f by its degree one Lagrange interpolant using the nodes $x_0 = a$ and $x_1 = b$.

b) Suppose that $[a, b]$ is divided into N equal subintervals, each of length $h = (b - a)/N$, using the $N + 1$ nodes $x_j = a + jh$ for $j = 0, 1, \ldots, N$. Using your result from part (a), give a closed formula for the composite quadrature on $[a, b]$. That is, a formula for

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{N} \alpha_i f(x_i).$$