Problem 1 Compute a full SVD of the matrix

\[
\begin{bmatrix}
0 & -\frac{3}{2} \\
3 & 0 \\
0 & \frac{15}{2}
\end{bmatrix},
\]

you may find the prime factorization \(234 = 2 \times 3^2 \times 13\) useful.

Problem 2 Solve the following problem using the QR factorization

\[
A = \begin{bmatrix}
0 & 4 & 1 \\
1 & 1 & 0 \\
0 & 3 & 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
10 \\
0 \\
-15
\end{bmatrix}.
\]

Problem 3 Find the quadratic polynomial minimizing the least square error

\[
\sum_{i=1}^{4}(p(x_i) - y_i)^2 \leq \min_{q(x) \in P_2} \sum_{i=1}^{4}(q(x_i) - y_i)^2
\]

for the \((x, y)\) data \{(0, 0), (1, 3/2), (2, -1), (3, -15/2)\}

Problem 4 Consider the problem of solving \(Ax = b\) where \(A\) is the symmetric matrix

\[
A = \begin{bmatrix}
2 & -5/4 & 0 \\
-5/4 & 11/4 & 9/7 \\
0 & 9/7 & 3/2
\end{bmatrix}.
\]

prove that the Jacobi method applied to solve this problem is convergent.
Problem 5 This problem has two parts.

a) Let \( f \) be a continuous function defined on the interval \([a, b]\). Derive the quadrature rule corresponding to approximating \( f \) by its first degree Lagrange interpolant using the nodes \( x_0 = a \) and \( x_1 = b \).

b) Prove that your quadrature rule from part (a) is exact to order \( m = 1 \).

Problem 6 State the equations for the explicit Runge-Kutta method

\[
\begin{array}{c|ccc}
0 & 0 & 0 \\
2/3 & 2/3 & 0 \\
1/4 & 3/4 \\
\end{array}
\]

for solving the initial value problem: \( y'(t) = f(t, y(t)) \), where \( y(0) = y_0 \) and \( 0 \leq t \leq 1 \), and show this method is of order two.

Problem 7 This problem has two parts. The system of nonlinear equations

\[
F(x) = 0, \quad \text{where } F(x) = F(x_1, x_2) = \begin{cases} 
2x_1 + x_2 - 1 \\
x_1^2 + x_2^2 - 1
\end{cases},
\]

has two solutions. Find one solution, \( \alpha \), of the system above.

a) Find a fixed point mapping \( G(x) \) for the system, above, and prove that there exists some neighborhood \( U \) of \( \alpha \) such that the fixed-point iteration

\[x^{(k+1)} = G(x^{(k)})\]

converges to \( \alpha \) if the initial iterate is chosen in \( U \).

b) Apply one step of Newton’s method, computing \( x^{(1)} \), using the initial iterate \( x^{(0)} = [0, \frac{3}{4}]^T \).

Problem 8 Consider the variable coefficient elliptic boundary value problem with homogeneous Dirichlet boundary conditions on \([0, 1]\) given by

\[
\begin{align*}
-\partial_x \left( (x + 1) \frac{\partial}{\partial x} u(x) \right) &= x & \text{(1a)} \\
u(0) = u(1) &= 0. & \text{(1b)}
\end{align*}
\]

The second order central differences for approximating the first and second derivatives are given by

\[
u''_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2), \quad u'_i = \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2).
\]

Determine the matrix system \( AU = F \) resulting from discretizing the system (1a)-(1b) using the second order finite differences, above, on the mesh \( T_h = \{0, 1/4, 1/2, 3/4, 1\} \). You may find it useful to use the product rule \((gu')' = gu'' + g'u'\).