Numerical Analysis Preliminary Examination
May 2024

Write your assigned number, the problem number and the page number on every page that you submit.

Submit **exactly 8 problems**, from the problems below, to be scored.

Problem 1 A matrix $A \in \mathbb{C}^{n\times n}$ is *unitary* if $A^*A = AA^* = I$ where A^* is the conjugate transpose of A and I is the $n \times n$ identity matrix. Prove that the singular values of a unitary matrix, A, are $\sigma_i = 1$, for $i = 1, 2, \ldots, n$.

Problem 2 Let $A \in \mathbb{R}^{n\times n}$. Prove that the Jacobi method for solving $A\mathbf{x} = \mathbf{b}$ is convergent if, for every $1 \le i \le n$, it holds that

$$\sum_{j\neq i}^{n} |A_{ij}| < |A_{ii}|$$

Problem 3 Take 5 steps of the power method to approximate the dominant eigenvalue, i.e. λ_1 satisfying $|\lambda_1| > |\lambda_2|$, of the matrix

$$\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix},$$

along with the approximation to the eigenvector corresponding to the dominant eigenvalue. Use $\mathbf{v}_0 = [1, 1]^T$ as your initial iterate. Show your work for every step.

Problem 4 This problem considers the use of a linear, stationary, first-order iterative method to find a vector $\hat{\mathbf{x}}$ that solves a matrix problem of the form $A\hat{\mathbf{x}} = \mathbf{b}$ where A is an invertible matrix and $\mathbf{b}$ is arbitrary, but fixed.

1. First, prove that any two $m \times n$ matrices, E and B, are equal if and only if $E\mathbf{v} = B\mathbf{v}$ for every $\mathbf{v} \in \mathbb{R}^n$
2. Suppose that $A \in \mathbb{R}^{n\times n}$ is an invertible matrix and that $\mathbf{b} \in \mathbb{R}^n$. Consider the iterative method

$$\mathbf{x}_{k+1} = M\mathbf{x}_k + N\mathbf{b},$$

   where M and N are $\mathbb{R}^{n\times n}$ matrices. Prove that if the iterative method, above, is convergent, then $M = I - NA$ as matrices (you may use your result from step 1 of this problem).

Problem 5 Consider the nonlinear system

$$x^3 = y^2$$
$$y - x = 4$$

The unique real-valued solution to the system of equations above is $(x, y) = (4, 8)$. Prove that the iteration

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} \sqrt[3]{y_k^2} \\ x_k + 4 \end{bmatrix},$$

converges to the solution $(\hat{x}, \hat{y}) = (4, 8)$ provided the initial iterate (x_0, y_0) is chosen sufficiently close to $(\hat{x}, \hat{y})$.

Problem 6 Compute an LU factorization of

$$\begin{bmatrix} 2 & 1 & -1 \\ -4 & 2 & 4 \\ 6 & 3 & 2 \end{bmatrix}$$

Problem 7 Consider the following matrix A and vector $\mathbf{b}$

$$A = \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -2 \\ -6 \\ 1 \\ -2 \\ 3 \end{bmatrix}$$

Find $\hat{\mathbf{x}} \in \mathbb{R}^2$ such that

$$||A\hat{\mathbf{x}} - \mathbf{b}||_2 = \min_{\mathbf{y} \in \mathbb{R}^2} ||\mathbf{y} - A\hat{\mathbf{x}}||_2,$$

where $||\mathbf{z}|| = \left(\sum_{i=1}^{N} z_i^2 \right)^{1/2}$ is the usual Euclidean norm.

Problem 8 Consider the interval $[0, \frac{\pi}{2}]$ partitioned into 3 subintervals of equal length. Apply the Cholesky factorization to solve the linear system that results from the use of (second order) central finite differences on the boundary value problem

$$-\partial_{xx} u = -sin(x), \quad u\,(0) = 0, \; u\left(\frac{\pi}{2}\right) = 1,$$

Problem 9 Given $\hat{x} \in [-1, 1]$, consider the problem of computing $\hat{y} \in [0, 1]$ such that $\hat{x}^2 + \hat{y}^2 = 1$ by using the algorithm

$$\hat{y} = \sqrt{1 - \hat{x}^2}$$

1. Show that the *relative condition number*, denoted by $\kappa(\hat{x})$, of this problem is unbounded for $\hat{x} \in (-1, 1)$
2. Suppose that $\hat{x}$ is such that $\kappa(\hat{x})$ is very large. Explain what can go wrong when computing $\hat{y}$, corresponding to such an $\hat{x}$, on a computing machine with finite precision.

Problem 10 In this problem you will construct a quadrature rule and use its composite form to estimate the integral of a given function.

1. Let $[a, b]$ be a fixed interval and consider the equally spaced interior nodes $x_0 = \frac{3a+b}{4}$ and $x_1 = \frac{a+3b}{4}$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and define the quadrature rule by

$$\int_a^b f(x)\,dx \approx \int_a^b \left[L_1(f)\right](x)\,dx$$

where $\left[L_1(f)\right](x)$ is the *linear Lagrange interpolant* of $f(x)$ with nodes x_0 and x_1. State a closed formula for your rule in the form

$$\int_a^b f\,(x)\;dx \approx \alpha_0 f\left(\frac{3a+b}{4}\right) + \alpha_1 f\left(\frac{a+3b}{4}\right),$$

that is, determine α_0 and α_1.

2. Use the composite quadrature, arising from the rule above, to estimate the integral of $f(x) = x^3$ on $[0, 1]$ with $N = 2$ equally spaced subintervals. That is, estimate

$$\int_0^1 x^3 \, dx = \int_0^{\frac{1}{2}} x^3 \, dx + \int_{\frac{1}{2}}^1 x^3 \, dx,$$

using the derived quadrature on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$.