Do all problems. For clarity and to assist the committee in grading, please write as neatly as possible to avoid any misunderstandings. You are allowed to use a single non-graphing, non-programmable calculator.

Condition numbers (50 pts)

1. Assume to know that $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the true solution to $\mathbb{A}x = b$, where

$$\mathbb{A} = \begin{bmatrix} 9.7 & 6.6 \\ 4.1 & 2.8 \end{bmatrix}, \quad b = \begin{bmatrix} 6.6 \\ 2.8 \end{bmatrix}.$$

a) Add a small perturbation of 0.0001 to the first component of b. Solve for the new x. **(20 pts)**
b) Rigorously justify why doing part a) is sensitive to perturbations in the vector b. **(30 pts)**

Linear spaces (40 pts)

2. Consider the L_2 norm and inner product with discrete measure. Recall that the discrete measure associated with the point set $\{t_1, t_2, \ldots, t_N\}$ is a measure $d\lambda$ that is nonzero only at t_i and has the value $w_i > 0$ there. For $u(t), v(t)$ having a finite L_2 norm, consider the following distance function:

$$d(u, v) = \frac{\|u - v\|_2}{1 + \|u - v\|_2}.$$

Prove that $d(\cdot, \cdot)$ defines a metric. However, it does not satisfy the absolute homogeneity property: $d(\alpha u, v) = |\alpha| \, d(u, v)$ for any $\alpha \in \mathbb{R}$. **(40 pts)**

Interpolation (70 pts)

3. Let $f(x)$ be a smooth function. Let $p_1(x)$ be the linear interpolation of $f(x)$ for $x_0 < x_1$ and $h = x_1 - x_0$.
a) Prove that $p_1^n(x) = \Pi_{i=1}^n p_1(x)$ is an interpolation polynomial of $f^n(x) = \Pi_{i=1}^n f(x)$ for any $n \geq 1$. **(10 pts)**
b) **(30 pts)** Assume that the actual value $f_\varepsilon(x_i)$ of $f(x_i)$ is given by

$$f(x_i) = f_\varepsilon(x_i) + \varepsilon_i, \quad i = 0, 1,$$

forming the corresponding $p_1^\varepsilon(x)$. Derive an error bound (in terms of h and $\varepsilon = \max\{|\varepsilon_i|\}$) of

$$\max_{x \in [x_0, x_1]} \left| f^2(x) - \left(p_1^\varepsilon(x)\right)^2 \right|.$$

Note that we denote $M = \max_{x \in [x_0, x_1]} (|f(x)| + |f''(x)|)$. Also, recall that

$$\max_{x \in [x_0, x_1]} |F(x) - p_1(F; x)| \leq \frac{h^2}{8} \max_{x \in [x_0, x_1]} |F''(x)|.$$

4. Prove or disprove the following claim: "Quadratic splines always produce a better approximation than linear splines". **(30 pts)**

Numerical differentiation (80 pts)

5.
a) **(40 pts)** Let $x_i = x_0 + ih$ $(i = -2, -1, 0, 1, 2)$. Assume that $f \in C^6(\mathbb{R})$. Apply the differentiation by interpolation method to derive the five-point midpoint formula: For $\xi = \xi(x_0)$ being some point in $(x_0 - 2h, x_0 + 2h)$,

$$f'(x_0) = D_h^{(1)} f(x_0) + \frac{h^4}{30} f^{(5)}(\xi),$$

where $D_h^{(1)} f(x_0) = \frac{1}{12h}[f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)]$.
b) **(10 pts)** Values of $f(x) = xe^x$ are given in the following table. Use the above five-point midpoint formula to approximate $f'(2)$. Then compute the absolute and relative errors.

x	1.8	1.9	2.0	2.1	2.2
$f(x)$	10.88	12.70	14.77	17.14	19.85

c) **(30 pts)** Assume that the actual value f_ε of f at x_i is given by $f(x_i) = f_\varepsilon(x_i) + \varepsilon_i$, where $\varepsilon_i \in (0, 1)$ is a noise value defined at each x_i. Define the (noisy) difference operator

$$D_{h,\varepsilon}^{(1)} f(x_0) = \frac{1}{12h}\left[f_\varepsilon(x_0 - 2h) - 8f_\varepsilon(x_0 - h) + 8f_\varepsilon(x_0 + h) - f_\varepsilon(x_0 + 2h)\right].$$

Let $\varepsilon = \max\{|\varepsilon_i|\}$ and $M = \max_{x \in [x_0 - 2h, x_0 + 2h]} \left|f^{(5)}(x)\right|$. Prove that the following error bound holds

$$\left|f'(x_0) - D_{h,\varepsilon}^{(1)} f(x_0)\right| \leq \frac{M}{30} h^4 + \frac{3\varepsilon}{2h}.$$

Numerical integration (60 pts)

6.

a) **(30 pts)** Using the fact that the two-point Gauss-Legendre quadrature formula is exact when $f \in \mathbb{P}_3$, show that it should have the following form:

$$\int_{-1}^{1} f(x)\, dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

b) Apply the formula to approximate the following integral:

$$I = \int_0^1 x e^x dx.$$

Then, compute the relative error. **(10 pts)**

7. (20 pts) Let $f \in C^{2n+2}[a,b]$. Consider the Gaussian quadrature formula associated with a weight function $w(x)$,

$$\int_a^b f(x)\, w(x)\, dx = G_n(f) + E_n(f),$$

$$G_n(f) = \sum_{i=0}^{n} w_i f(x_i), \quad E_n(f) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \int_a^b \prod_{i=0}^{n} (x - x_i)^2\, w(x)\, dx$$

for some $\xi \in (a,b)$. Assume that the actual value of x_i is $x_i^{\varepsilon} \in (a,b)$ satisfying $|x_i - x_i^{\varepsilon}| \leq \varepsilon$ for $\varepsilon \in (0,1)$ and for any $i = \overline{0,n}$. Accordingly, we have w_i^{β} that is the actual value of w_i, and it satisfies that $\left|w_i - w_i^{\beta}\right| \leq \beta$ for $\beta \in (0,1)$ and for any $i = \overline{0,n}$. Then, the actual value of the Gaussian quadrature formula is given by

$$G_{n,\varepsilon,\beta}(f) = \sum_{i=0}^{n} w_i^{\beta} f(x_i^{\varepsilon}).$$

Prove that

$$\left|\int_a^b f(x)\, w(x)\, dx - G_{n,\varepsilon,\beta}(f)\right| \leq M \int_a^b w(x)\, dx \left[\frac{(b-a)^{2n+2}}{(2n+2)!} + \varepsilon\right] + \beta(n+1) M (\varepsilon + 1).$$

Note that $\int_a^b w(x)\, dx = \sum_{i=0}^{n} w_i$, and you may find $M = \max_{x\in[a,b]} \left\{|f(x)|, |f'(x)|, \left|f^{(2n+2)}(x)\right|\right\}$ useful.

Approximations of nonlinear equations (40 pts)

8. Let p be a unique fixed point of $C^1 \ni g : [a,b] \to [a,b]$ with $|g'(x)| \le k < 1$. For $p_0 \in [a,b]$ being any initial guess, consider the fixed-point iterations $p_n = g(p_{n-1})$.
a) **(20 pts)** Prove that

$$|p_n - p| \le \frac{k^n}{1-k} |p_1 - p_0| .$$

You may find the inequality $|x-y| \ge |z-y| - |x-z|$ useful.
b) **(20 pts)** Let $p_0^\varepsilon \in (a,b)$ be the actual value of p_0 satisfying $|p_0^\varepsilon - p_0| \le \varepsilon$ for $\varepsilon \in (0,1)$. Then, we get the corresponding iterations $p_n^\varepsilon = g\left(p_{n-1}^\varepsilon\right)$. Prove that

$$|p_n^\varepsilon - p| \le k^n \left(\varepsilon + \frac{1}{1-k} |p_1 - p_0| \right).$$

Initial value problems (60 pts)

9. Let $t_i = ih = \frac{iT}{N}$ be the equally distributed mesh points of the time interval $[0,T]$. Consider the Euler's method,

$$\begin{cases} Y_{i+1} = Y_i + hf(t_i, Y_i), \\ Y_0 = y_0, \end{cases}$$

to approximate $y : [0,T] \to [a,b]$ satisfying $y'(t) = f(t,y)$ with $y(0) = y_0$. Assume that f satisfies the Lipschitz condition,

$$|f(t,y_1) - f(t,y_2)| \le L |y_1 - y_2| \quad \text{for any } (t,y_1), (t,y_2) \in [0,T] \times [a,b],$$

and $y \in C^2[0,T]$ with $M = \max_{t\in[0,T]} |y''(t)| > 0$. Here, $[a,b]$ is an arbitrarily large domain that contains both y and $\{Y_i\}$.
a) **(10 pts)** Let $\Phi_h(y(t_i)) = y(t_i) + hf(t_i, y(t_i))$. Using the Taylor expansion, prove that

$$|y(t_{i+1}) - \Phi_h(y(t_i))| \le \frac{h^2 M}{2}.$$

b) **(20 pts)** Using the Lipschitz property of f, prove that

$$|\Phi_h(y(t_i)) - Y_{i+1}| \le (1+hL) |y(t_i) - Y_i| .$$

Then, show that

$$\max_{1\le i\le N} |y(t_i) - Y_i| \le \frac{hM}{2L} \left(e^{LT} - 1\right).$$

You may find the inequality $1 + x \le e^x$ for $x \ge 0$ useful.
c) **(30 pts)** Let the Euler polygon Y_h be defined as

$$Y_h(t) = Y_i + (t - t_i) f(t_i, Y_i) \quad \text{for } t_i \le t \le t_{i+1}.$$

Let Y_i^ε be the actual value of Y_i such that $|Y_i^\varepsilon - Y_i| \le \varepsilon_i$ for $\varepsilon_i \in (0,1)$. Let $\varepsilon = \max\{\varepsilon_i\}$. Then, let $Y_h^\varepsilon(t)$ be the Euler polygon associated with Y_i^ε. Prove that

$$\max_{1\le i\le N} \max_{t\in[t_i,t_{i+1}]} |y(t) - Y_h^\varepsilon(t)| \le \frac{hM}{2L} \left(e^{LT} - 1\right)(1+hL) + \frac{h^2 M}{2} + (1+hL)\varepsilon.$$