ODE Preliminary Exam

August 2022

Part A Answer 3 of the following 4 questions. Clearly indicate which 3 you choose.

1. Consider the linear system $\dot{x} = Ax$, where

$$A = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (a) Determine the fundamental matrix.

   (b) Use the method of variation of parameters to solve the linear nonhomogeneous system $\dot{x} = Ax + b(t)$, where $x(0) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $b(t) = \begin{bmatrix} \sin(2t) \\ \cos(2t) \\ 1 \end{bmatrix}$.

2. Consider the system

$$\begin{aligned} \dot{x} &= 4y^3 - xy^2 \\ \dot{y} &= -3x + x^2 y. \end{aligned}$$

   (a) Find all equilibrium points.

   (b) Apply LaSalle's Invariance Principle to determine the stability of the origin.

3. Consider the equation

$$\dot{x} = x - \mu x(1 - x),$$

   where parameter $\mu > 0$.

   (a) Identify all equilibria as functions of μ and classify their stability.

   (b) Identify any bifurcations that take place as μ is varied. Sketch a bifurcation diagram.

4. Consider the system

$$\begin{aligned} \dot{x} &= \mu x - y - x^3 \\ \dot{y} &= x + \mu y - y^3. \end{aligned}$$

   (a) Classify the stability of the equilibrium at the origin as a function of μ.

   (b) Does this system exhibit a Hopf bifurcation as μ is varied? Provide evidence. *Hint:* $\frac{1}{2} \leq \cos^4(\theta) + \sin^4(\theta) \leq 1$.

Part B Answer the following question.

1. Consider the system

$$\dot{x} = y - x^2$$
$$\dot{y} = \alpha x + \beta - y.$$

   (a) Identify all equilibria and use linearization to investigate their stability as they depend on parameters α and β.

   (b) Identify and describe any bifurcations that take place as α and β are varied. Provide sample phase-planes.

   (c) Identify regions of $\alpha - \beta$ parameter space that yield qualitatively different behaviors (i.e. quantify the number of equilibria in different regions of $\alpha - \beta$ parameter space).