

ODE Preliminary Exam

August 2023

Part A Answer 3 of the following 4 questions. Clearly indicate which 3 you choose.

1. The following ODE represents an adaptive control (y-variable) being applied to a control variable x :

$$\begin{aligned}\frac{d}{dt}x(t) &= -(y - a)x \\ \frac{d}{dt}y(t) &= \gamma x^2\end{aligned}\tag{1}$$

where a, γ are constants. Prove that all trajectories converge to the line $\{x = 0\}$.

2. Prove that the origin is stable for the following ODE :

$$\begin{aligned}\frac{d}{dt}x(t) &= 2xy^2 - x^3 \\ \frac{d}{dt}y(t) &= -2y^3 + \frac{1}{9}x^2y\end{aligned}\tag{2}$$

3. Consider the 2nd order ODE

$$\frac{d^2}{dt^2}x(t) - (\mu - x(t)^2)\frac{d}{dt}x(t) + x(t) = 0.\tag{3}$$

Describe the fixed points, periodic cycles, and bifurcations taking place as the parameter μ is changed over the range $(-1, 1)$.

4. What is meant by the ω -limit set of a point ? Prove that if the orbit of a point x has a bounded orbit, then its ω -limit set must be a compact, connected and invariant set.

Part B Answer the following question.

1. We study the van der Pol oscillator, which is described by the second order ODE

$$\frac{d^2}{dt^2}x(t) + \mu(x(t)^2 - 1)\frac{d}{dt}x(t) + x(t) = 0,\tag{4}$$

with the parameter μ varying over the range $(3, 5)$.

- (a) Rewrite this equation as a first order ODE in \mathbb{R}^2 .
- (b) Describe the stability of the origin for different values of μ .
- (c) Prove that there can be at most one periodic cycle for the system, for any $\mu \in (3, 5)$.
- (d) Prove that there is one periodic cycle for the system, with help of the Hopf bifurcation theorem.