ODE Preliminary Exam
August 2025

Part A Answer all questions in this section.

1. Consider a linear, time-varying ODE in $\mathbb{R}^d$:

$$\frac{d}{dt}x = A(t)x. \tag{1}$$

   (a) [5] Prove that the solution to (1) with initial conditions $x(t_0) = x_0$ can be written as

$$x(t) = \Phi_A(t_0, t)x_0,$$

   where $\Phi_A(t_0, t)$ is a matrix valued function of the start time t_0 and present time t.

   (b) [10] Now suppose that $A(t)$ is T-periodic. Prove that

$$\Phi_A(0, t) = \Phi_A(T, t + T), \quad \forall t \in \mathbb{R}.$$

2. [15] Investigate the stability of the origin for the following ODE, using the center manifold theorem

$$\begin{aligned} \frac{d}{dt}x_1 &= -x_1^2 \\ \frac{d}{dt}x_2 &= x_1^2 - x_2 \end{aligned} \tag{2}$$

   Provide the full statement of any theorem that you use. You do not need to prove the theorem.

3. Consider the ODE

$$\begin{aligned} \frac{d}{dt}x_1 &= 2x_1 + x_1^2 \\ \frac{d}{dt}x_2 &= -x_2 \end{aligned} \tag{3}$$

   (a) [3] Describe the stability properties of the fixed point at the origin.

   (b) [2] Define the global and local stable sets of the fixed point.

   (c) [3] Express the global stable set in terms of the local stable manifold.

   (d) [12] Use the iterative method with up to 3 iterations, to estimate the stable manifold.

4. Consider the following ODE in $\mathbb{R}^2$ expressed in polar coordinates

$$\frac{d}{dt}r = 4r^2 - 2r^3 + \mu r. \tag{4}$$

   Here μ is a parameter that can vary.

   (a) [4] Find all fixed points and periodic cycles for various parameter values, and describe their stability,

   (b) [8] Identify and characterize all the bifurcations taking place.

   (c) [3] Sketch a bifurcation diagram.

Part B Answer any one from the following set :

5. Consider the ODE

$$\begin{aligned}\frac{d}{dt}x_1 &= -x_2 - \mu x_2^2 x_1 \\ \frac{d}{dt}x_2 &= x_1\end{aligned} \tag{5}$$

in which μ is a parameter.

(i) [4] Classify the stability of all fixed points, for various values of μ.

(ii) [4] Explain with reason whether a Hop bifurcation occurs, and where.

(iii) [12] Determine the stability of the fixed point for $\mu > 0$.

6. Consider the following ODE in $\mathbb{R}^2$ expressed in polar form :

$$\begin{aligned}\frac{d}{dt}r &= -2r^3 + r(1.1 + \cos\theta) \\ \frac{d}{dt}\theta &= 1\end{aligned} \tag{6}$$

(i) [5] Find all fixed points and periodic cycles for the system. You should also eliminate the possibility of fixed points and periodic cycles other than the ones you list.

(ii) [3] Characterize the stability of the origin

(iii) [4] Prove that all trajectories of this ODE remain bounded.

(iv) [8] Use the Poincare Bendixson theorem and your previous observations to characterize the stability of all the periodic cycles that you have found.