ODE Preliminary Exam
May 2025

Instructions Please read the instructions carefully.

- You are not allowed to use any electronic devices, print or online media.
- There are two parts in the Question paper. Attempt all problems from Part A, and one from Part B.
- Please start answering a new question on a new page.
- In case you attempted more than one question in Part B, please specify, on the top page of your answer sheet, which question we should grade. If you failed to specify, we will grade the first attempted answer for Part B, and delete the last.

Part A Answer all questions in this section.

1. Consider a general parametric IVP in $\mathbb{R}^d$

$$\frac{d}{dt}x(t) = V(x(t), \mu), x(0) = x_0. \tag{1}$$

with μ being a parameter picked from $\mathbb{R}$.

(a) [5] State in precise terms what it means for solutions to be *locally* continuous with respect to x_0 and with respect to μ.

(b) [5] State without proof the result on local continuity of solutions to ODEs with respect to initial conditions. Frame the statement in terms of a general non-parametric ODE, and use a variable other than ”x” to denote the state-variable.

(c) [5] Use this theorem to prove that the solutions to (1) depends continuously on μ

2. [15] Investigate the stability of the origin for the following ODE, using the center manifold theorem

$$\begin{aligned} \frac{d}{dt}x_1 &= x_1(x_2 - x_1) \\ \frac{d}{dt}x_2 &= x_1 - x_2 + \alpha x_1^2 + x_1 x_2 \end{aligned} \tag{2}$$

Provide the full statement of any theorem that you use. You do not need to prove the theorem.

3. Consider the ODE

$$\begin{aligned} \frac{d}{dt}x_1 &= x_1 + x_2^2 \\ \frac{d}{dt}x_2 &= -x_2 - x_1^2 \end{aligned} \tag{3}$$

(a) [3] Describe the stability properties of the fixed point at the origin.

(b) [2] Define the global and local stable sets of the fixed point.

(c) [3] State without proof the local stable manifold theorem.

(d) [12] Use the iterative method with up to 3 iterations, to estimate the stable manifold.

4. Consider the following ODE in $\mathbb{R}$ modelling a certain population dynamics.

$$\frac{d}{dt}x = \alpha x\left(1 - \frac{x}{M}\right) - \mu x. \tag{4}$$

Here α, M are fixed, positive constants, and μ is a parameter that can vary.

(a) [4] Find all fixed points for various parameter values, and describe their stability,

(b) [8] Identify and characterize all the bifurcations taking place.

(c) [3] Sketch a bifurcation diagram.

Part B Answer any one from the following section :

5. Consider a general time-invariant linear ODE

$$\frac{d}{dt}x = V(x) = Ax \tag{5}$$

(i) [2] What is the Jacobian DV of V at the origin ?

(ii) [2] Suppose $DV(\vec{0})$ is invertible. Is it hyperbolic ? Explain with reason.

We assume henceforth that the origin is a hyperbolic fixed point.

(iii) [6] Prove that the stable and unstable sets $W^s(\vec{0})$ and $W^u(\vec{0})$ of the origin are linear subspaces.

(iv) [10] Prove that $W^s(\vec{0})$ and $W^u(\vec{0})$ are complimentary subspaces.

6. Consider the linear, time-varying ODE in $\mathbb{R}^d$:

$$\frac{d}{dt}x = A(t)x. \tag{6}$$

Suppose $u^{(1)}, \ldots, u^{(d)}$ are a collection of d independent initial conditions.

(i) [8] Suppose for each $j = 1, \ldots, d$, $u_j(t)$ is a solution of the ODE (6) with initial condition $u^{(j)}$. Arrange these solution vectors as columns of a matrix :

$$\mathcal{U}(t) := \left[\left(\ u_1(t)\ \right), \left(\ u_2(t)\ \right), \cdots, \left(\ u_d(t)\ \right)\right].$$

Prove that $\mathcal{U}(t)$ is invertible for every time t.

(ii) [12] Suppose that $A(t)$ is T-periodic. Prove that the matrix $\mathcal{U}(t)^{-1}\mathcal{U}(t+T)$ is time t-independent.