ODE Preliminary Exam

August 2023

Part A. Answer 3 of the following 4 questions. Clearly indicate which 3 you choose.

1. The following ODE represents an adaptive control (y-variable) being applied to a control variable x :

$$\begin{aligned} \frac{d}{dt}x(t) &= -(y-a)x \\ \frac{d}{dt}y(t) &= \gamma x^2 \end{aligned} \tag{1}$$

   where a, γ are constants. Prove that all trajectories converge to the line $\{x = 0\}$.

2. Prove that the origin is stable for the following ODE :

$$\begin{aligned} \frac{d}{dt}x(t) &= 2xy^2 - x^3 \\ \frac{d}{dt}y(t) &= -2y^3 + \frac{1}{9}x^2 y \end{aligned} \tag{2}$$

3. Consider the 2nd order ODE

$$\frac{d^2}{dt^2}x(t) - (\mu - x(t)^2)\frac{d}{dt}x(t) + x(t) = 0. \tag{3}$$

   Describe the fixed points, periodic cycles, and bifurcations taking place as the parameter μ is changed over the range $(-1, 1)$.

4. What is meant by the ω-limit set of a point ? Prove that is the orbit of a point x has a bounded orbit, then its ω-limit set must be a compact, connected and invariant set.

Part B. Answer the following question.

1. We study the van der Pol oscillator, which is described by the second order ODE

$$\frac{d^2}{dt^2}x(t) + \mu\left(x(t)^2 - 1\right)\frac{d}{dt}x(t) + x(t) = 0, \tag{4}$$

   with the parameter μ varying over the range $(3, 5)$.

   (a) Rewrite this equation as a first order ODE in $\mathbb{R}^2$.

   (b) Describe the stability of the origin for different values of μ.

   (c) Prove that there can be at most one periodic cycle for the system, for any $\mu \in (3, 5)$.

   (d) Prove that there is one periodic cycle for the system, with help of the Hopf bifurcation theorem.