ODE Preliminary Exam

May 2023

Part A. Answer 3 of the following 4 questions. Clearly indicate which 3 you choose.

1. Consider the system

$$\begin{aligned} \frac{d}{dt}x(t) &= -x^3 + y^5 \\ \frac{d}{dt}y(t) &= -2x^3 - y^3 \end{aligned} \tag{1}$$

   Prove that the origin is globally asymptotically stable.

2. Consider the system

$$\begin{aligned} \frac{d}{dt}x(t) &= (x-y)(y-1) + y^2 + 2y \\ \frac{d}{dt}y(t) &= 2y + (x-y)y \end{aligned} \tag{2}$$

   Find local stable and unstable manifolds for the origin.

3. Consider the system

$$\begin{aligned} \frac{d}{dt}x(t) &= x(ax + by + c) \\ \frac{d}{dt}y(t) &= y(dx + ey + f) \end{aligned} \tag{3}$$

   where a, b, c, d, e, f are real constants. Explain with justification, whether this system has any periodic orbits or not.

4. State (without proof) the local existence theorem for solutions to ODE with continuous vector fields. Use this to prove that the maximal time domain of existence of solutions must be an open interval.

Part B. Answer the following question.

1. In this problem we try to build our theoretical understanding of the behavior of an ODE in the vicinity of a hyperbolic fixed point. So consider an ODE on $\mathbb{R}^d$ of the form

$$V(x) = Ax + G(x)$$

   where $A \in \mathbb{R}^{d\times d}$ is a hyperbolic matrix, and $G : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1 function satisfying

$$G(\vec{0}) = \vec{0}, DG(\vec{0}) = [0]_{d\times d}.$$

   Assume that the hyperbolic matrix A has non-trivial stable and unstable directions E^s and E^u respectively,

(a) Define what is meant by the global and local stable set of the fixed point at $\vec{0}$.

(b) Express (without proof) the local stable manifold as the fixed point of an operator.

(c) Prove why this formulation implies that the local stable set must be a manifold.

(d) Express the global stable manifold in terms of the local stable manifold.