Ordinary Differential Equations
Prelim Examination Aug 5, 2024

You are not allowed to use any calculators, computers or cellphones

There are five questions in all. Attempt both questions from Part A (30% each question) and any two questions from Part B (20% each question)

Please start answering a new question from a new page.

In case you attempted more than two questions in Part B, please specify, on the top page of your answer sheet, which two questions we should grade. If you failed to specify, we will grade the first two answers for Part B, in the order attempted, and delete the last.

Part A Attempt both the questions from this part A.

[**Question 1:**] Let A and B be two $n \times n$ matrices with real numbers. They are called similar if there exists a non-singular matrix P, with real numbers, such that

$$P^{-1}AP = B.$$

(a) Answer the following questions by saying yes or no, supported by a reason.

i. If A and B are two similar matrices, they must have the same set of eigenvalues.

ii. If A and B are two similar matrices, they must have the same determinant.

iii. If A and B are two similar matrices, they must have the same trace.

iv. If A has real and distinct eigenvalues, then A is similar to a diagonal matrix.

(b) i. Argue if the following two matrices are similar

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

ii. If

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

and

$$B = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Argue if A and B are similar. If A and B are similar, find a matrix P such that

$$P^{-1}AP = B.$$

If A and B are not similar, say why.

[**Question 2:**]

(a) Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 3x_1 - 5x_2 - x_1^2 x_2 - x_1^3. \end{aligned}$$

Using Bendixson's criterion, show that there does not exist any periodic orbit.

(b) Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= x_1 + x_2 - 2x_1 x_2. \end{aligned}$$

Using index method, argue if it is possible to have a closed orbit $\mathcal{O}$ that would encircle the points $(0, 0)$ and $(1, 1)$.

(c) Consider the 2^{nd} order equation

$$\ddot{\theta} = -3\cos\theta - 5\dot{\theta}. \quad (1)$$

Define variables

$$x_1 = \theta,\ x_2 = \dot{\theta}$$

i. Write down state equations in terms of x_1 and x_2.

ii. Calculate all the equilibrium points of (1).

iii. Calculate the Jacobian matrix at each of the equilibrium points. Compute the corresponding eigenvalues.

iv. Characterize the equilibrium points. Are they stable/unstable, node/spiral/saddle etc?

Part B Attempt any two questions from this part B. If you attempt more than 2 questions, clearly indicate which two questions should be graded. Else, we will grade 2 questions in the order attempted.

[**Question 3:**] Consider the well known pendulum equation in state variables described as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10 \, \sin x_1. \end{aligned} \tag{2}$$

i. Define Lyapunov function

$$V(x_1, x_2) \;=\; 10\,(1 - \cos x_1) \;+\; \frac{1}{2}\, x_2^2 \tag{3}$$

and argue the stability properties of (2), using this Lyapunov function (3). Clearly state the domain.

ii. Let us now modify the pendulum equation by adding a friction, described as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10 \, \sin x_1 \;-\; 6\, x_2. \end{aligned} \tag{4}$$

Using the Lyapunov function (3) argue the stability properties of (4), clearly stating the domain.

iii. Changing the Lyapunov function to

$$V(x_1, x_2) \;=\; 10\,(1 - \cos x_1) \;+\; \frac{1}{2}\,(x_1 \;\; x_2) \begin{pmatrix} p_{11} \; p_{12} \\ p_{12} \; p_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{5}$$

find numerical values of p_{11}, p_{12}, p_{22} to ensure that (4) is asymptotically stable, using the Lyapunov function (5). Is this asymptotic stability local or global?

iv. State Lasalle's theorem and analyze the stability properties of (4) using the function $V(x_1, x_2)$ described in (3).

[**Question 4:**] Consider the following system of equation

$$\begin{aligned} \dot{x} &= -y - x^3 \\ \dot{y} &= x^5. \end{aligned} \tag{6}$$

Show that for the system (6), the equilibrium $(0, 0)$ is asymptotically stable using the following Lyapunov function

$$V(x, y) \;=\; x^6 + xy^3 + 3y^2. \tag{7}$$

Hint: Using Young's Inequality, one can use the following:

$$|xy^3| \;\le\; \frac{1}{6}\, x^6 \;+\; \frac{5}{6}\, y^2, \;\text{ if }\; |y| \;\le\; 1.$$

$$|-x^3y^3| \;\le\; \frac{3}{8}\, x^8 \;+\; \frac{5}{8}\, y^4, \;\text{ if }\; |y| \;\le\; 1.$$

$$|3x^6y^2| \;\le\; \frac{9}{4}\, x^8 \;+\; \frac{3}{64}\, y^4, \;\text{ if }\; |y| \;\le\; \frac{1}{2}.$$

[**Question 5:**] Consider the following system of equation

$$\begin{aligned} \dot{V} &= 10\,(V - \tfrac{V^3}{3} - R + 1.5) \\ \dot{R} &= .8\,(-R + 1.25V + 1.5). \end{aligned} \tag{8}$$

(a) What are the two isocline equations

$$\frac{dV}{dt} = 0 \text{ and } \frac{dR}{dt} = 0,$$

on the (R, V) plane? (Answer this question by drawing a rough sketch on the plane.)

(b) Intersect the two isocline equations to calculate the equilibrium points.

(c) Calculating the two eigenvalues of the corresponding Jacobian matrix at the equilibrium points, conclude the type of the equilibrium points. (is it stable/unstable, node/spiral/saddle etc)

(d) We want to construct, if possible, an asymptotically stable limit cycle, enclosed by a rectangular boundary as shown in the attached Fig. 1, around the equilibrium point (The corners of the rectangle are points A,B,C,D). State (without proof) the Poincare-Bendixon throrem and argue how this theorem is applicable in ensuring such a limit cycle.

(e) Starting from the point A with coordinates $(3, \frac{21}{4})$, on the isocline

$$\frac{dR}{dt} = 0,$$

find other corners of the rectangle as shown in the Fig. 1. Assume that the V coordinate of the points B and C are V_0 (no need to calculate V_0 numerically.).

(f) Argue why the trajectories starting on the open line segment AB must move downward and to the left. Argue why the trajectories starting on the open line segment BC must move downward and to the right. Similar statements can be made regarding segments CD and DA.

(g) Complete the argument that there must exist inside the rectangular domain ABCD at least one asymptotically stable limit cycle.

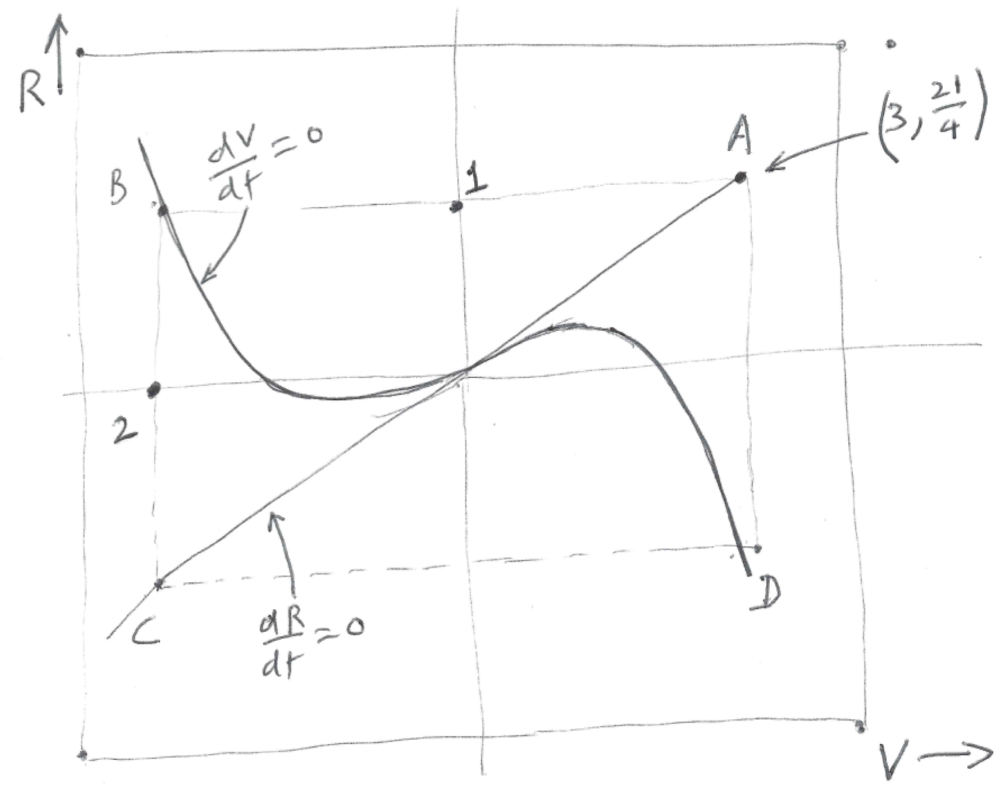

Fig 1: This figure corresponds to Question 5, part e. A has co-ordinates $\left(3, \frac{21}{4}\right)$, on the isocline $\frac{dR}{dt} = 0$. The point B is on the isocline $\frac{dV}{dt} = 0$, the point C is on the isocline $dR/dt = 0$.

segments AB and CD are parallel to the V axis, BC and AD are parallel to the R axis.