Ordinary Differential Equations
Prelim Examination May 2024

You are not allowed to use any calculators, computers or cell-phones

There are five questions in all. Attempt both questions from Part A (30% each question) and any two questions from Part B (20% each question)

Please start answering a new question from a new page.

In case you attempted more than two questions in Part B, please specify, on the top page of your answer sheet, which two questions we should grade. If you failed to specify, we will grade the first two answers for Part B, in the order attempted, and delete the last.

Part A Attempt both the questions from this part A.

[**Question 1:**] Let A be the matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) Writing A as

$$A = \Gamma + N,$$

where

$$\Gamma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Argue whether the following equality holds true or not

$$e^{At} = e^{\Gamma t} e^{Nt}.$$

(b) Calculate (if possible) a nonsingular matrix P such that

$$P^{-1}AP = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Otherwise argue why such a P would not exist.

(c) Writing

$$A_1 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

calculate $e^{A_1 t}$, using any method.

(d) Writing

$$e^{At} = \alpha_0(t)I + \alpha_1(t)A + \alpha_2(t)A^2,$$

calculate $\alpha_0(t)$, $\alpha_1(t)$, $\alpha_2(t)$.

[**Question 2:**] In this problem we are analyzing the Vander Pol oscillator

$$\ddot{x} \;+\; \mu\,\dot{x}(x^2 - 1) \;+\; x \;=\; 0. \tag{1}$$

We define the cubic function

$$F(x) = \frac{1}{3}x^3 - x.$$

(a) Using state variables x, w where

$$w \;=\; \dot{x} \;+\; \mu F(x),$$

show that (1) can be written as

$$\begin{aligned} \dot{w} \;&=\; -x \\ \dot{x} \;&=\; w \;-\; \mu F(x). \end{aligned} \tag{2}$$

(b) Rescale w by μ and define a new variable

$$y \;=\; \frac{1}{\mu}w.$$

Show that (2) can be written as

$$\begin{aligned} \dot{y} \;&=\; -\tfrac{1}{\mu}\,x \\ \dot{x} \;&=\; \mu\,[y \;-\; F(x)]. \end{aligned}$$

(c) Consider the graph of $y = F(x)$ as shown in $Fig.$ 1, on the last page. The segments AD and BC are parallel to the x-axis.

i. Calculate coordinates of the points A, B, C and D.

ii. Using qualitative reasoning, argue that for large values of μ, the phase diagram of the Vander Pol oscillator would settle down to the green curve shown in $Fig.$ 1.

iii. Argue that the time it takes to move from D to A and B to C is negligibly small for large values of μ.

iv. Estimate the time it takes for the state to move from A to B on the curve $y = F(x)$. Hence estimate the time period of oscillation for the Vander Pol oscillator.

Hint:

$$T_{AB} \;=\; \int_{x_A}^{x_B} \frac{1}{\dot{w}}\,\frac{dw}{dx}\,dx$$

where
T_{AB} : Time for the state to move from A to B,
x_A : Coordinate of the point A,
x_B : Coordinate of the point B,
$\mu y \;=\; w \;=\; \mu F(x)$ on the path joining A to B.

Part B Attempt any two questions from this part B. If you attempt more than 2 questions, clearly indicate which two questions should be graded. Else, we will grade 2 questions in the order attempted.

[**Question 3:**] Euler equations for a rotating rigid spacecraft are given by

$$\begin{aligned} J_1\dot{\omega}_1 &= (J_2 - J_3)\,\omega_2\omega_3 + u_1 \\ J_2\dot{\omega}_2 &= (J_3 - J_1)\,\omega_3\omega_1 + u_2 \\ J_3\dot{\omega}_3 &= (J_1 - J_2)\,\omega_1\omega_2 + u_3, \end{aligned}$$

where $\omega_1, \omega_2, \omega_3$, are the components of the angular velocity vector ω along the principal axes, $u_1,\ u_2,\ u_3$, are torque inputs applied about the principal axes and $J_1,\ J_2,\ J_3$, are the principal moments of inertia, $J_i > 0,\ i = 1, 2, 3$.

i. Show that with $u_1 = u_2 = u_3 = 0$, the origin $\omega = 0$ is stable. Is it asymptotically stable?

ii. Suppose the torque inputs apply the feedback control

$$u_i = -k_i\,\omega_i,$$

where $k_1,\ k_2,\ k_3$ are positive constants. Show that the origin of the closed loop system is globally asymptotically stable.

[**Question 4:**] Investigate the stability of the origin by using center manifold theorem

$$\begin{aligned} \dot{x}_1 &= -x_2^2 \\ \dot{x}_2 &= -x_2 + x_1^2 + x_1x_2. \end{aligned}$$

[**Question 5:**] Consider the 2^{nd} order system

$$\begin{aligned} \dot{x} &= -x^5 + y^7 \\ \dot{y} &= -2x^5 - y^3. \end{aligned}$$

Prove that the origin is globally asymptotically stable.

Hint: Try a Lyapunov function of the form

$$V(x, y) = ax^n + by^m$$

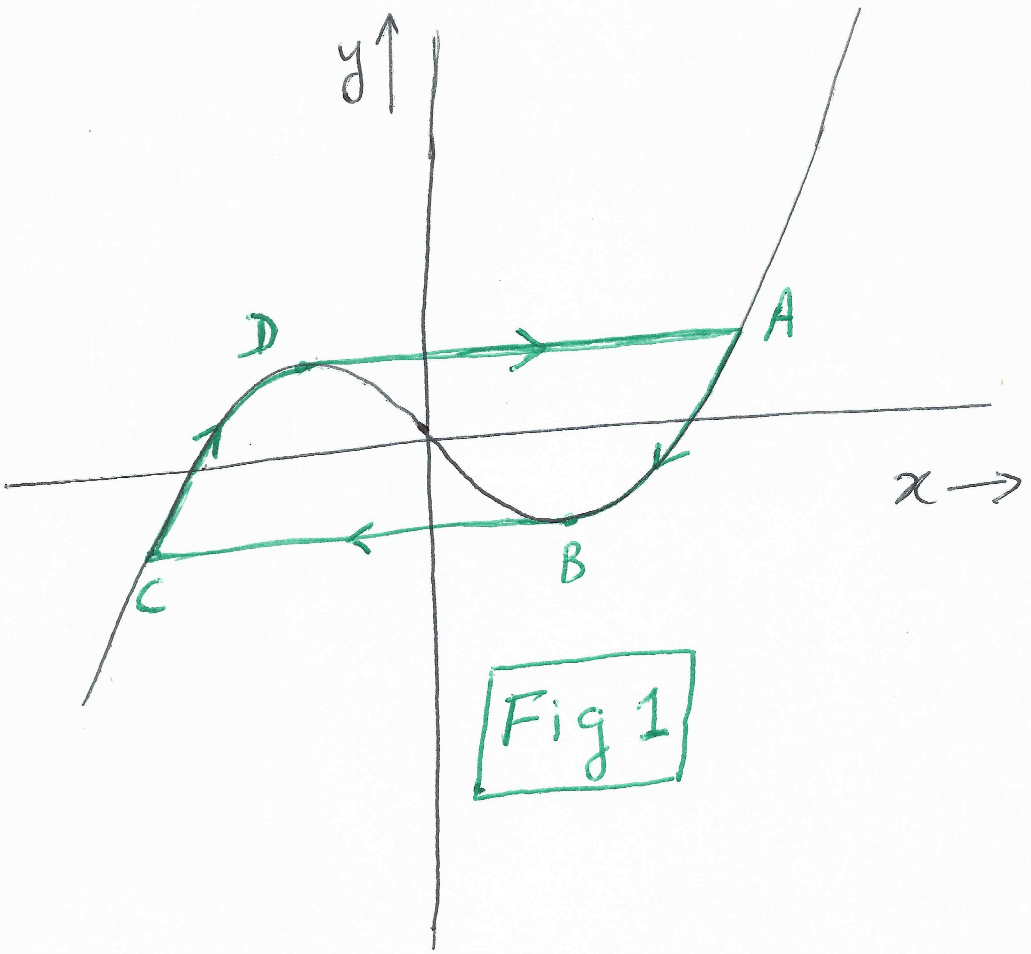

Fig 1