1996 Preliminary Examination in Ordinary and Partial Differential Equations

Do 3 problems in Part I and 3 of the problems from Part II. You must clearly indicate which 6 problems are to be graded.

Part I.

Problems **1** and **2** refer to the following linear systems

$$x'(t) = A(t)x(t), \qquad (LH)$$

$$x'(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0, \qquad (LNH)$$

where $x(t), f(t) \in R^n$, $A(t)$ is a real $n \times n$ matrix, and $A(t), f(t)$ are continuous on an open interval I that contains t_0.

1. (a) Define what is meant by a fundamental matrix of (LH), explain why it exists, and derive a formula for a solution of the initial value problem (LNH).
(b) Prove that the unique solution of (LNH) exists on the whole interval I, whether it be finite or infinite.

2. In (LH) suppose that $A(t)$ is constant and given by

$$A = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ -\gamma_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & -\gamma_2 & \alpha_2 \end{bmatrix}.$$

(a) If $\alpha_1 = \alpha_2 = 0$ and $\gamma_1 = \beta_1$ and $\gamma_2 = \beta_2$, show that all solutions of (LH) are periodic if β_1/β_2 is rational.
(b) Suppose that $\gamma_1 = \gamma_2 = \alpha_1 = \beta_2 = 0$ and $\beta_1 = \alpha_2 = 1$. True or False: All solutions of (LH) are unbounded. Explain your answer by proof or counter-example.

3. (a) Suppose $u_1(x)$ is a solution of

$$y'' + g_1(x)y = 0$$

and $u_2(x)$ is a solution of

$$y'' + g_2(x)y = 0.$$

If g_1, g_2 are continuous functions and $g_2(x) > g_1(x)$, prove that between any two consecutive zeros of $u_1(x)$ there is a zero of $u_2(x)$.
(b) Let $y(x)$ be a solution of $y'' + r(x)y = 0$ where $r(x) > kx^{-2}$ for some $k > 1/4$. Show that $y(x)$ has infinitely many positive zeros.

4. (a) If $k > 0$ show that $x = 0$ is stable for

$$x'' + kx' + \omega^2 x + \beta x^3 = 0.$$

(b) Show that $x = 0$ is asymptotically stable for

$$x'' + kx' + (1 + \frac{1}{1+t^2})x = 0.$$

Part II.

5. (a) Solve the initial value problem

$$uu_x + u_t = 0, \quad u(x,0) = f(x).$$

(b) If $f(x) = x$, show that the solution exists for all $t > 0$.
(c) If $f(x) = -x$, show that a shock develops, that is, the solution blows up in finite time.

6. (a) Define what is meant by a Green's function for the boundary value problem (BVP)

$$\Delta u = 0, \quad x \in \Omega,$$

$$u(x) = f(x), \quad x \in \partial\Omega.$$

(b) Construct the Green's function for the BVP when Ω is the upper half-plane in R^2, $\{(x,y) : y > 0\}$.
(c) If $f(x)$ is bounded and continuous, show that the function defined by

$$u(x,y) = \begin{cases} \dfrac{y}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{f(t)}{(t-x)^2 + y^2}\, dt & y > 0 \\ f(x) & y = 0 \end{cases}$$

is harmonic in the upper half-plane and continuous for $y \geq 0$

7. Suppose $u(x,t)$ is a solution of

$$\begin{aligned} \Delta u &= u_{tt}, & x \in R^n,\ t > 0 \\ u(x,0) = u_0(x),&\ u_t(x,0) = u_1(x) & x \in R^n \end{aligned}$$

where u_0, u_1 are smooth functions with compact support.
(a) If $n = 3$, show that for each x there exists a time $T(x)$ such that if $t > T(x)$, $u(x,t) = 0$.
(b) If $n = 2$, show that for each x

$$\lim_{t \to \infty} u(x,t) = 0.$$

(c) If $n = 1$, show that for each x there exists a time $T(x)$ such that if $t > T(x)$, $u(x,t)$ is constant.

8. (a) Separate variables to show that the solution of

$$u_t = u_{xx}, \quad 0 < x < \pi,\ t > 0$$

$$u(x,0) = f(x), \quad 0 \leq x < \pi$$

$$u(0,t) = 0 = u(\pi,t), \quad t \geq 0$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx)$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

Assume that $f(x)$ is continuous, piecewise smooth and $f(0) = f(\pi) = 0$.
(b) Show that for each $t > 0$ the function $u(x,t)$ defined by this series represents a C^∞ function in x and satisfies the the heat equation.
(c) Prove that $u(x,t)$ is continuous on $[0,\pi] \times [0,\infty)$ and that

$$\lim_{t \to 0} u(x,t) = f(x), \quad x \in [0,\pi].$$

(d) Suppose that $u(1,t) = 0$ for all t. Show that $u(x,t) = 0$ for all x, t.