PRELIMINARY EXAMINATION IN DIFFERENTIAL EQUATIONS
MAY, 1997

Instructions: Do three (3) problems from Part I and three (3) problems from Part II. You *must* indicate which problems are to be graded.

Part I. ODE

1. A special case of the Lotka-Volterra equations that describe a predator-prey model are given by

$$u_1' = -u_1 - 2u_1^2 + u_1u_2$$
$$u_2' = -u_2 + 7u_1u_2 - 2u_2^2,$$

where $u_1(t)$ and $u_2(t)$ denote populations at time t.

(a) Find the biologically meaningful critical points (also known as equilibrium points or fixed points).

(b) Show that if the populations $u_1(0)$ and $u_2(0)$ at time $t = 0$ are sufficiently small, then both species will become extinct. Illustrate by describing (or drawing) the phase portrait for the linearized system.

(c) Show that one biologically meaningful critical point corresponds to a saddle point for the linearized system.

2. (a) For the system

$$x' = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} x = Ax, \qquad \omega > 0$$

compute e^{At}.

(b) Use the result of Part (a) and the variation of parameters formula to show that the general solution of

$$\frac{d^2y}{dt^2} + \omega^2 y = f(t)$$

is given by

$$y(t) = c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{1}{\omega}\int_0^t \sin(\omega(t-s))f(s)\,ds.$$

9. Consider the initial boundary value problem for a function $u = u(x,t)$ of two real variables

$$u_{xx} - u_{tt} - au_t - bu = 0, \qquad 0 < x < \ell,\ t > 0$$
$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x),$$
$$u(0,t) = 0, \quad u_x(\ell,t) = 0, \qquad t \geq 0.$$

where $a, b > 0$ are constants.

(a) Show that if u is a solution of the initial boundary value problem,

$$(2u_t u_x)_x - (u_x^2 + u_t^2 + bu^2)_t - 2au_t^2 = 0.$$

Hint: multiply the differential equation by $2u_t$.

(b) Prove that if u satisfies the initial boundary value problem, then

$$\int_0^\ell (u_x^2 + u_t^2 + bu^2)|_{t=T}\, dx \leq \int_0^\ell (u_x^2 + u_t^2 + bu^2)|_{t=0}\, dx.$$

(c) State and prove a uniqueness theorem for the above initial boundary value problem.

10. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Consider the boundary value problem

$$\Delta u + c(x)u = f(x), \qquad x \in \Omega$$
$$u = g(x), \qquad x \in \Gamma_1$$
$$\frac{\partial u}{\partial n} + k(x)u = h(x), \qquad x \in \Gamma_2,$$

where Γ_1 and Γ_2 are smooth surfaces such that $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. It is assumed that k, g, and h are continuous functions on $\partial\Omega$ and that $c, f \in C(\bar{\Omega})$.

Show that if $c \leq 0$ and $k \geq 0$, a solution $u \in C^2(\bar{\Omega})$ of the boundary value problem is unique, if it exists (i.e., there is at most one such solution u).

Show that all solutions of the system

$$x'(t) = [A + C(t)]x(t)$$

are asymptotically stable.

Part II. PDE

6. Solve the following Cauchy Problem, where $u = u(x, y)$ is a function of two real variables:

$$uu_x + yu_y = x$$
$$u(x, 1) = 2x.$$

7. Consider the following PDE in two variables:

$$y^2 u_{xx} + yu_{xy} + u_{yy} = u_x. \qquad (*)$$

Show that the curve C parameterized by $(x(s), y(s)) = (s, s)$ for $s \in (-\infty, \infty)$ is non-characteristic for equation $(*)$.

Let u be the solution of equation $(*)$ for the Cauchy data

$$u(s, s) = f(s)$$
$$\frac{\partial u}{\partial n}(s, s) = 0.$$

Determine the values of $u_x(s, s)$, $u_y(s, s)$, $u_{xx}(s, s)$, $u_{xy}(s, s)$ and $u_{yy}(s, s)$.

8. (a) Consider the following initial boundary value problem for a function $u = u(x, t)$ of two real variables:

$$u_{tt} = u_{xx}, \qquad 0 < x < \pi,\ t > 0$$
$$u_x(0, t) = 0, \quad u_x(\pi, t) = 0, \qquad t \geq 0$$
$$u(x, 0) = f(x), \quad u_t(x, 0) = 0$$

Find a formal solution of the problem by the method of separation of variables.

(b) Assuming that f is C^4 and that f satisfies $f'(0) = 0$, $f'(\pi) = 0$, $f'''(0) = 0$ and $f'''(\pi) = 0$, show carefully that the formula you derived in part (a) of the problem defines a classical solution of the initial boundary value problem.

3. Consider the two dimensional system

$$x' = y + \lambda x(x^2 + y^2)$$
$$y' = -x + \lambda y(x^2 + y^2)$$

(a) Show that the origin is a stable fixed point for the linearized system.
(b) Show that the origin is an unstable fixed point for the full system, if $\lambda > 0$.

4. (a) Find the eigenvalues and eigenfunctions for the boundary value problem

$$y'' + \lambda y = 0, \qquad 0 < x < \pi$$
$$y(0) = 0$$
$$y'(\pi) = 0.$$

(b) Find the Green's function for this boundary value problem in the case $\lambda = 0$.
(c) Use the Green's function from part (b) of the problem to solve the boundary value problem

$$y'' = x^2, \qquad 0 < x < \pi$$
$$y(0) = 0$$
$$y'(\pi) = 0.$$

5. For this problem, it will be helpful to recall Gronwall's Inequality, which states the following: if f_1, f_2 and p are continuous real valued functions on $[a, b]$, $p \geq 0$, and

$$f_1(t) \leq f_2(t) + \int_a^t p(s) f_1(s)\, ds, \qquad t \in [a, b]$$

then

$$f_1(t) \leq f_2(t) + \int_a^t p(s) f_2(s) \exp\left[\int_s^t p(u)\, du\right] ds, \qquad t \in [a, b].$$

Assume that A is an $n \times n$ matrix and that all of the eigenvalues of A have real part strictly less than 0. Let C be a continuous matrix valued function on $[0, \infty)$ such that

$$\int_0^\infty \|C(s)\|\, ds < \infty.$$