# 1998 Preliminary Examination in Ordinary and Partial Differential Equations

Do 3 problems in Part I and 3 of the problems from Part II. You must clearly indicate which 6 problems are to be graded.

## Part I.

**1.** Consider the following linear systems

$$x'(t) = A(t)x(t), \qquad (LH)$$

$$x'(t) = A(t)x(t) + f(t), \qquad (LNH)$$

where $x(t), f(t) \in \mathbb{R}^n$, $A(t)$ is a real $n \times n$ matrix, and $A(t), f(t)$ are continuous on an open interval $I$ that contains $t_0$.
(a) Prove that the set of all solutions of $(LH)$ forms an n-dimensional vector space.
(b) Define what is meant by a fundamental matrix of $(LH)$ and explain why it exists.
(c) Assume that a fundamental matrix is known and derive the variation of parameters formula for the solution of $(LNH)$ subject to the initial condition $x(t_0) = x_0$.

**2.** Consider the Sturm-Liouville problem,

$$(xy'(x))' + \frac{\lambda}{x} y(x) = 0, \quad y(1) = y(e) = 0.$$

(a) Show that the eigenvalues and eigenfunctions are given by

$$\lambda_n = n^2\pi^2, \quad y_n(x) = \sin(n\pi \ln x), \quad n = 1, 2, 3, \cdots$$

(b) Construct the Green's function for this problem if $\lambda = 1$.
(c) Consider the nonhomogeneous problem

$$(xy'(x))' + \frac{\pi^2}{x} y(x) = \sin(n\pi \ln x), \quad y(1) = y(e) = 0.$$

For which integers $n$ is this problem solvable?

**3.** Consider the Bessel equation

$$y''(x) + \left(1 + \frac{1 - 4p^2}{4x^2}\right) y(x) = 0$$

(a) If $0 \le p < 1/2$, show that every nontrivial solution of the Bessel equation has at least one zero in every interval of length $\pi$.
(b) If $p = 1/2$, show the zeros of every solution are separated by an interval of length $\pi$.
(c) If $p > 1/2$, show that every solution can have at most one zero in any interval of length $\pi$.

**4.** (a) If $g(0) = 0$ and $xg(x) > 0$ show that $x(t) = 0$ is a stable equilibrium for

$$x''(t) + g(x(t)) = 0.$$

(b) Determine a condition on $\alpha$ so that $x(t) = 0$ is a stable equilibrium for

$$x''(t) + \left[1 + \frac{t}{(1+t)^\alpha}\right] x(t) = 0.$$

## Part II.

**5.** Consider the quasilinear differential equation

$$uu_x + u_y = 1$$

subject to the initial condition

$$u(x,x) = 0, \quad x \in \mathbb{R}.$$

Solve for $u(x,y)$.

**6.** (a) Define what is meant by a Green's function for the boundary value problem (BVP)

$$\Delta u = 0, \;\; x \in \Omega \subset \mathbb{R}^n,$$

$$u(x) = f(x), \;\; x \in \partial\Omega.$$

(b) Construct the Green's function for the BVP when $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$ and use it to show that the solution to the boundary value problem may be written as

$$u(x) = \frac{1-|x|^2}{4\pi} \int_{|y|=1} \frac{f(y)}{|x-y|^3} d\sigma_y.$$

(c) Give conditions on $f$ so that this formula for $u(x)$ provides a solution that is in $C^2(\Omega) \cap C^0(\bar{\Omega})$.

**7.** Suppose $u(x,t)$ is a solution of

$$\begin{array}{ll} \Delta u = u_{tt}, & x \in \mathbb{R}^3, \; t > 0 \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x) & x \in \mathbb{R}^3 \end{array}$$

where $u_0, u_1$ are smooth functions with compact support.
(a) Show that for each $x$ there exists a time $T(x)$ such that if $t > T(x), \; u(x,t) = 0$.
(b) Prove that $\sup_{x \in \mathbb{R}^3} u(x,t) \leq \frac{M}{t}$ for a constant $M$.

**8.** (a) Separate variables to construct a series solution of

$$u_t = u_{xx}, \;\; 0 < x < \pi, \; t > 0$$

$$u(x,0) = x, \;\; 0 \leq x < \pi$$

$$u_x(0,t) = 0 = u_x(\pi,t), \;\; t \geq 0$$

(b) Carefully justify that the series solution satisfies the boundary conditions and the initial condition and show that for each $t > 0$ the function $u(x,t)$ defined by this series represents a $C^\infty$ function in $x$ that satisfies the heat equation.