1999 ODE/PDE PRELIMINARY EXAM

Do 3 problems from Part I and 3 problems from Part II. You must clearly indicate which 6 problems are to be graded.

PART I: ODE

1. (Gronwall's inequality and continuous dependence for ODEs)

   a) Let γ be a constant, k a nonnegative constant and h a continuous function on a finite interval $[a,b]$. Prove that

   $$h(t) \leq \gamma + k \int_a^t h(s)\,ds, \quad \forall\, t \in [a,b] \quad \Rightarrow \quad h(t) \leq \gamma\, e^{k(t-a)}\ \forall\, t \in [a,b].$$

   b) Let $f(t,x)$ be continuous in t and x and Lipschitz with respect to x, with Lipschitz constant K, on a domain D in $\mathbb{R}^2$. Let φ and ψ be solutions of $y' = f(t,y)$, respectively, such that $\varphi(0) = \varphi_0$, $\psi(0) = \psi_0$, existing on a common interval $|t| \leq \alpha < \infty$. Prove that

   $$|\varphi_0 - \psi_0| \leq \delta, \quad \Rightarrow \quad |\varphi(t) - \psi(t)| \leq \delta e^{K\alpha}.$$

2. Consider the Initial Value Problem (IVP) for the n-th order ordinary differential equation on $\mathbb{R}$

   $$y^{(n)}(t) + a_1(t) y^{(n-1)}(t) + \cdots + a_n(t) y(t) = f(t)$$
   $$y(t_0) = y_1,\ y^{(1)}(t_0) = y_2,\ \cdots,\ y^{(n-1)}(t_0) = y_n,$$

   with f and $a_j \in C(\mathbb{R})$ for $j = 1, \cdots, n$.

   a) Write the equation as a first order system $\begin{cases} \dfrac{d}{dt}x(t) = A(t)x(t) + B(t) & (*) \\ x(t_0) = x_0. \end{cases}$

   where $x(t) \in \mathbb{R}^n$. Carefully explain why, for every $x_0 \in \mathbb{R}^n$, a solution to $(*)$ exists for all $t \in \mathbb{R}$.

   b) Let $\Phi(t)$ be a fundamental matrix for $(*)$ with $f = 0$. Use Φ to derive the variation of parameters formula for $(*)$.

3. Determine the stability/instability of the origin for the nonlinear systems. State carefully any theorems that you use to establish your results.

   a) $\begin{cases} x' = -x - y - 3x^2 y \\ y' = -2x - 4y + y\sin(x) \end{cases}$

   b) $\begin{cases} x' = -y + x^3 \\ y' = x - y^3 \end{cases}$

4. Given the Boundary Value Problem (BVP) $\begin{cases} y'' + y = -f(x) \\ y(0) = 0, \ y(\ell) = 0 \end{cases}$

   a) Find a value of ℓ (i.e., a number) so that the Green's function exists. For this ℓ, construct the Green's function and give a formula (using the Green's function) for the solution of the BVP.

   b) Under what conditions on ℓ does a Green's function not exist? Give an ℓ for which the Green's function does not exist and give a condition on f that will guarantee that a solution of the BVP exists.

PART II: PDE

1. Consider the first order quasi-linear initial value problem

$$\begin{aligned} c(u)u_x + u_t &= 0 \\ u(x,0) &= f(x) \end{aligned}$$

   where f and c are smooth functions and $u = u(x,t)$ with $x \in \mathbb{R}$ and $t \geq 0$.

   a) Show that for sufficiently small t the solution is defined implicitly by

$$u(x,t) = f(x - c(u)t).$$

   b) Show that if the functions f and c are both nonincreasing or both nondecreasing the solution exists for all $t \geq 0$. (That is, no shocks develop.)

   c) In the general case, find a formula for the "breaking time" t_b (i.e., the first time at which a shock develops).

2. Find the explicit solution to the nonhomogeneous IVP

$$\begin{aligned} &u_{tt} = u_{xx} + x, \quad x \in \mathbb{R}, \quad t > 0, \\ &u(x,0) = x^2, \quad u_t(x,0) = 0. \end{aligned}$$

3. Consider the Cauchy problem,

$$\begin{aligned} &u_{tt} = \Delta u, && x \in \mathbb{R}^n, \quad t > 0, \\ &u(x,0) = u_0(x), && u_t(x,0) = u_1(x), \end{aligned}$$

   where $u_0, u_1 \in C^\infty(\mathbb{R}^n)$ with compact support. For $n = 1$ write D'Alembert's formula and for $n = 3$ write Kirchoff's formula. Explain why Huygen's principle is valid for $\mathbb{R}^3$ and not for $\mathbb{R}$.

4. Do both parts a) and b).

   a) Consider the Dirichlet Problem

   $$\Delta u = 0, \quad x \in \Omega; \quad u(x) = f(x), \quad x \in \partial\Omega, \tag{DP}$$

   where Ω is a bounded smooth domain in $\mathbb{R}^n$. Let f_1 and f_2 be two functions defined and $C^2(\partial\Omega)$. Let u_1 and u_2 be $C^2(\overline{\Omega})$ solutions of (DP) corresponding to f_1 and f_2, respectively. Prove that for any $\epsilon > 0$, if

   $$|f_1(x) - f_2(x)| \leq \epsilon, \quad \text{for all } \ x \in \partial\Omega,$$

   then

   $$|u_1(x) - u_2(x)| \leq \epsilon, \quad \text{for all } \ x \in \overline{\Omega}.$$

   State carefully any theorems that are used in the proof.

   b) Determine whether the Dirichlet Problem in $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \,:\, y > 0\}$

   $$\Delta u(x, y) = 0, \quad (x, y) \in \mathbb{R}^2_+, \quad u(x, 0) = x, \quad x \in \mathbb{R} \tag{DP}$$

   has a unique solution. Prove or give a counterexample.

5. Consider the following nonhomogeneous boundary value problem for the heat equation,

   $$\begin{aligned} &u_t = u_{xx}, \quad 0 < x < 1, \ t > 0 \\ &u(x, 0) = 0, \quad 0 < x < 1 \\ &u(0, t) = 0, \quad u(1, t) = 1, \ t \geq 0. \end{aligned} \tag{BVP}$$

   a) Let $u(x, t) = v(x, t) + x$. Find an appropriate heat problem for v, solve the resulting problem for v and thus find the solution u of (BVP).

   b) Determine the steady state solution of (BVP), i.e., find $\varphi(x)$ satisfying

   $$\varphi(x) = \lim_{t \to \infty} u(x, t).$$