2000 ODE/PDE PRELIMINARY EXAM

Do 3 problems in Part I and 3 problems in Part II. You must clearly indicate which 6 problems are to be graded.

PART I: ODE

1. Consider the first order nonlinear system of ordinary differential equations for two unknown functions $x_1 = x_1(t)$ and $x_2 = x_2(t)$, $t \in \mathbb{R}$

$$x_1' = x_2 \sin(x_1^2 - 2x_1x_2) + 2$$
$$x_2' = x_1\sqrt{1 + \cos^2(x_1 + x_2)} + 5x_2 \exp(-x_1^2).$$

   a) Show that the associated initial value problem at time $t_0 = 0$ has a unique solution on some sufficiently small time interval $(-\alpha, \alpha)$. Give a precise statement of the general theorem you need to demonstrate the above result and explain why all the conditions of this theorem are satisfied.

   b) Limiting your consideration to the time interval $[0, \infty)$, derive a global in time (i.e., in t) a priori estimate for the solution of the above problem.

   c) Use the estimate derived in part *b)* to prove that there exists a unique solution to the problem on the entire time interval $[0, \infty)$. Give a precise statement of any general theorems you need to carry out this proof.

2. Consider the general first order homogeneous linear system:

$$\boldsymbol{x}'(t) = A(t)\boldsymbol{x}(t), \qquad (*)$$

   where $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $A(\cdot)$ is a continuous, $n \times n$ matrix valued function of $t \in \mathbb{R}$.

   a) Prove that the set $\mathcal{F}$ of all solutions of the system $(*)$ is an n-dimensional vector space over $\mathbb{R}$. In your proof you may assume existence and uniqueness of the solutions to $(*)$ for all $t \in \mathbb{R}$ and for all initial conditions.

   b) Give definitions of a *fundamental system of solutions* and a *fundamental matrix* for $(*)$. Write the differential equation satisfied by the fundamental matrix. State (without proof) the variation of parameters formula for the solution of the initial value problem

$$\boldsymbol{x}'(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t),$$
$$\boldsymbol{x}(0) = \boldsymbol{x}_0.$$

   c) Assume that $A(t) = A$ does not depend on t. What is the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = I$ (the identity matrix), in this case? Write a power series formula for $\Phi(t)$. State the group property of $\Phi(t)$.

   d) Find the fundamental matrix $\Phi(t)$ for $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$. Clearly show all steps of your calculation.

3. Consider the system of ordinary differential equations for two unknown functions $x = x(t)$ and $y = y(t)$:

$$\begin{aligned} x' &= y(x-a) \\ y' &= -x(y-b) \end{aligned} \tag{NLS}$$

where a and b are real constants.

a) Find all stationary states (critical points) of the system (NLS).

b) Find the linearization of the system (NLS) about each critical point and discuss the stability/instability of the linearized system in the following two cases:

   1) $a > 0,\ b > 0$;
   2) $a > 0,\ b < 0$.

c) Use the method of linearization (and your results from part b)) to investigate the stability/instability of the critical points for (NLS) in the two cases treated in part b).

   If for some critical point this method does not lead to any conclusion, so state. Give precise statements of the general theorems you need to establish your results.

d) In both of the above cases sketch the phase portrait of the system near the critical points. If for some critical point the linearization method does not provide a definite phase portrait, sketch all possible types of phase portraits in the vicinity of this point.

4. Consider the Sturm-Liouville operator $(Lu)(x) = -\big(p(x)u'(x)\big)' + q(x)u(x),\ x \in [a,b]$ defined on the domain

$$\mathcal{D}(L) = \left\{u \in C^2[a,b]\ :\ \alpha_1 u(a) + \alpha_2 u'(a) = 0,\ \beta_1 u(b) + \beta_2 u'(b) = 0\right\},$$

where $p \in C^1[a,b]$, $q \in C[a,b]$, $p(x) > 0$ and the parameters in the boundary conditions satisfy $\alpha_1^2 + \alpha_2^2 \neq 0$, $\beta_1^2 + \beta_2^2 \neq 0$.

a) Write the formula for the solution of the equation

$$Lu - \lambda u = f, \quad u \in \mathcal{D}(L)$$

   (here $f \in C[a,b]$ is a given function) in terms of the Green's function of the operator $(L - \lambda I)$ (here I is the identity operator).

   Describe the values of the parameter λ for which the Green's function fails to exist and describe how they are distributed in the real line.

   What equation and boundary conditions does the Green's function satisfy?

b) Describe (without proof) the algorithm for the construction of the Green's function and give a formula for this function.

c) Find the Green's function for the operator

$$\big[(L - \lambda I)u\big](x) = -u''(x) - \lambda u(x), \quad x \in [0,1]$$

   with the boundary conditions

$$u'(0) = 0, \quad u(1) = 0.$$

   Investigate two cases: 1) $\lambda > 0$, 2) $\lambda = 0$. Find the values of λ for which the Green's function does not exist.

PART II: PDE

1. The general first order quasilinear partial differential equation with two independent variables is given by

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u) \tag{2}$$

where $u = u(x,y)$.

a) Write the system of characteristic equations for (2) and define the characteristic curves.

b) State a theorem (without proof) that gives a characterization of the solutions of the above partial differential equation in terms of the relation between the graph of the solution and the characteristic vector field, $\boldsymbol{a}(x,y,u) = \big(a(x,y,u),\, b(x,y,u),\, c(x,y,u)\big)$.

c) Consider the following initial value problem for a first order quasilinear partial differential equation in the two variables (x,t) for the unknown function $u = u(x,t)$:

$$\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = \frac{1}{u}, \quad x > 0,\ t > 0,$$

$$u(x,0) = \sqrt{x}.$$

i) Verify that the initial condition is noncharacteristic.

ii) Find the solution. Clearly indicate all steps of your calculations.

2. Consider the Cauchy problem:

$$\begin{aligned} &u_{tt} - c^2\Delta u = f(x,t), \quad x \in \mathbb{R}^n,\ t > 0 \\ &u(x,0) = u_0(x), \\ &u_t(x,0) = u_1(x). \end{aligned} \tag{IVP}$$

a) Write (without proof) the fomula for the solution of the problem (IVP) in two cases: $n = 2$ and $n = 3$.

b) Assume that u_0 and u_1 have compact support and $f \equiv 0$. Sketch a picture and use it to explain why Huygen's principle holds for $n = 3$.

c) Use the Kirchhoff formula to find the solution of the problem (IVP) in the case when

$$x = (x_1,x_2,x_3) \in \mathbb{R}^3, \quad f(x,t) = 0, \quad u_0(x) = 0, \quad \text{and} \quad u_1(x_1,x_2,x_3) = x_1 + x_2.$$

Clearly show all steps of your calculations.

3. a) Define the fundamental solution for the Laplace operator in space dimensions $n = 2$ and $n = 3$.

   b) State (without proof) the *fundamental formula* for the Laplace operator.

   c) Give the definition of the Green's function for the problem

$$\begin{aligned} &\Delta u = 0, \quad u = u(x), \quad x \in \Omega \subset \mathbb{R}^3, \\ &u(x) = g(x), \quad x \in \partial\Omega \end{aligned} \tag{BVP}$$

   (Ω is a bounded domain with smooth boundary).

   d) Write the fomula for the solution of (BVP) in terms of the Green's function. Derive this formula using the *fundamental formula* for the Laplace operator. You are allowed to use the Green's identity without proof.

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $\partial\Omega$. Assume that $p \in C^1(\overline{\Omega})$, $q \in C(\overline{\Omega})$, $p(x) > 0$ for all $x \in \Omega$ and $\alpha \in C(\partial\Omega)$.

   a) Consider the following elliptic differential operator

$$L\varphi = -\nabla \cdot \big(p(x)\nabla\varphi\big)(x) + q(x)\varphi(x), \quad x \in \Omega, \tag{EO}$$

   defined on the domain

$$\mathcal{D}(L) = \left\{ \varphi \in C^2(\overline{\Omega}) \, : \, \left(\frac{\partial \varphi}{\partial \nu} + \alpha(x)\varphi \right) \bigg|_{x \in \partial\Omega} = 0 \right\}.$$

   Give a precise statement of the spectral theorem for the elliptic operator L in (EO).

   b) Consider the initial boundary value problem

$$\begin{aligned} &u_t + Lu = 0, \quad u = u(x,t) \quad x \in \Omega, \ t > 0, \\ &\left(\frac{\partial u}{\partial \nu} + \alpha(x)u \right) \bigg|_{\partial\Omega} = 0, \\ &u(x,0) = u_0(x). \end{aligned} \tag{IBP}$$

   Based on the results from part a) derive a formula for the solution to (IBP) using the eigenfunction expansion method.

   c) Apply the method of parts a) and b) to find the solution of the following problem for the 1-dimensional heat equation:

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \quad x \in [0,1], \ t > 0, \\ &u'(0) = 0, \quad u'(1) = 0, \\ &u(x,0) = u_0(x) = \begin{cases} 0 & \text{if } \ 0 \le x \le 1/2 \\ 1 & \text{if } \ 1/2 < x \le 1 \end{cases}. \end{aligned}$$