2001 ODE/PDE PRELIMINARY EXAM

DO 3 PROBLEMS FROM PART I AND 3 PROBLEMS FROM PART II. YOU MUST CLEARLY INDICATE WHICH 6 PROBLEMS ARE TO BE GRADED.

PART I: ODE

1. Consider the 2nd order nonlinear system of ordinary differential equations for two unknown functions $x_1 = x_1(t)$ and $x_2 = x_2(t)$, $t \in \mathbb{R}$

$$\begin{aligned} x_1'' &= 4x_1 - x_2 e^{x_1 x_2} \\ x_2'' &= 10x_2 - x_1 e^{x_1 x_2}. \end{aligned} \tag{S}$$

   a) As a preliminary step to part b), write the system as a first order system. For this new system give the general form of the initial conditions at $t = 0$. What is the state space for the system (S)?

   b) Show that the above initial value problem has a unique solution on some sufficiently small time interval $(-\epsilon, \epsilon)$. **Give a precise statements of any theorems you need to establish this result and explain why all the conditions of these theorems are satisfied**.

   c) Represent the system (S) in the form of a Newtonian equation of motion

$$\boldsymbol{x}''(t) + \nabla U(\boldsymbol{x}(t)) = 0, \tag{N}$$

   where $\boldsymbol{x}(t) = \big(x_1(t), x_2(t)\big)^T \in \mathbb{R}^2$ and $U(\boldsymbol{x}) = U(x_1, x_2)$ is an appropriate potential energy function.

   d) Use (N) and your explicit expression for $U(x_1, x_2)$ to derive global in time (i.e., for $t \in [0, \infty)$) a priori estimate for the solution of the above initial value problem.

2. Consider the first order linear nonhomogeneous system of ordinary differential equations

$$\begin{aligned} x_1' &= x_1 + 2x_2 + 1 \\ x_2' &= x_2 + 2x_3 \\ x_3 &= x_3 + e^t \end{aligned} \tag{NHS}$$

   with initial conditions

$$x_1(0) = 1, \quad x_2(0) = -1, \quad x_3(0) = 0. \tag{IC}$$

   a) Write the system (NS) in the matrix form

$$\frac{d}{dt}\boldsymbol{x}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}(t) \tag{1}$$

   where $\boldsymbol{x} = \big(x_1(t), x_2(t), x_3(t)\big)^T \in \mathbb{R}^3$. Find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = I$ (identity matrix) for the corresponding homogeneous system

$$\frac{d}{dt}\boldsymbol{x}(t) = A\boldsymbol{x}(t) \tag{HS}$$

b) Write the fundamental system of solutions and the general solution for (HS). Describe (without proof) the set of all fundamental matrices for (HS).

c) State the variation of parameters formula for the above initial value problem (NHS), (IC). Use this formula and an explicit expression for $\Phi(t)$ to find the solution of the problem (NHS), (IC).

3. Consider the system of ordinary differential equations for two unknown functions $x = x(t)$ and $y = y(t)$:

$$\begin{aligned} x' &= -x^3 + y, \\ y' &= x - y. \end{aligned} \tag{S}$$

a) Find all stationary states (critical points) of the system (S).

b) Find the linearization of the system (S) about each critical point and discuss the stability/instability of all the resulting linearized systems.

c) Use the method of linearization (and your results from part b)) to investigate the stability/instability of the critical points for (S). **Give precise statements of any theorems you need to establish your results**.

d) Sketch the phase portrait of the system (S) near each of the critical points.

4. Consider the Sturm-Liouville operator $(Au)(x) = -\big(x^2 u'(x)\big)'$, defined on the domain

$$\mathcal{D}(A) = \big\{u \in C^2[1,2] \,:\, u(1) - u'(1) = 0,\ u(2) - 2u'(2) = 0\big\}.$$

a) Find the Green's function for this operator.

b) Write the formula for the solution of the boundary value problem

$$Au = f, \qquad u \in \mathcal{D}(A)$$

(here $f \in C[1,2]$ is a given function) in terms of the Green's function. Use this formula to calculate the solution for $f(x) = x$.

c) Consider the above operator A on the domain

$$\mathcal{D}(A) = \big\{u \in C^2[1,a] \,:\, u(1) - u'(1) = 0,\ u(a) - 2u'(a) = 0\big\}.$$

Find the value of a ($a > 1$) for which the Green's function for A does not exist.

PART II: PDE

1. Consider the following linear 2nd order partial differential equation for unknown function $u = u(x,y)$:

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

a) Classify the equation and reduce it to the appropriate canonical form. **Clearly show all steps of your calculations**.

b) Integrate the canonical form to obtain the general solution.

2. Consider the Cauchy problem:

$$\begin{aligned} &u_{tt} - c^2 \Delta u = f(x,t), \quad x \in \mathbb{R}^n, \ t > 0 \\ &u(x,0) = u_0(x), \\ &u_t(x,0) = v_0(x). \end{aligned} \tag{IVP}$$

a) Write (without proof) the formula for the solution of the problem (IVP) in two cases: $n = 3$ and $n = 1$ (in the later case $\Delta u = u_{xx}$).

b) Let $n = 1$, $f(x,t) = 0$ and assume that the functions u_0 and v_0 vanish outside the interval $[a,b]$. Show that for any x there exists a $T(x) > 0$ such

$$u(x,t) = \text{constant} \quad \text{ for all } \quad t \geq T(x).$$

Find the value of the above constant.

c) Find the solution of the problem (IVP) in the case $n = 3$ with

$$\begin{aligned} &u_0(x) = \boldsymbol{a} \cdot x, \quad (\boldsymbol{a} = (a_1, a_2, a_3)^T \in \mathbb{R}^3 \quad \text{is a given vector}), \\ &v_0(x) = 0, \\ &f(x,t) = \|x\|^2 \, t, \end{aligned}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$.

3. Consider the boundary value problem

$$\begin{aligned} &-\Delta u(x) = f(x), \quad x \in \Omega \subset \mathbb{R}^n, \ (n = 2,3) \\ &u(x) = g(x), \quad x \in \partial\Omega \end{aligned} \quad . \tag{BVP}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ is the boundary of Ω (assumed to be piecewise smooth).

a) State (**completely but without proof**) a theorem which provides an algorithm for construction of the Green's function for the problem (BVP). Write the formula for the solution of (BVP) in terms of the Green's function.

b) Use the method of images to construct the Green's function for (BVP) when Ω is the half ball $\mathbb{R}^2$, i.e. assume that

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \ : \ |x| < R, \ x_2 > 0\}.$$

(Here $|x|$ is the Euclidean norm of x.) **Clearly explain why the function you have constructed is indeed the Green's function**.

4. Consider the following initial boundary value problem for the heat equation with nonhomogeneous boundary conditions:

$$\begin{aligned} &u_t = u_{xx}, \quad u = u(x,t), \quad x \in (0,1), \quad t > 0, \\ &u(0,t) = 0, \quad u_x(1,t) = t \\ &u(x,0) = x. \end{aligned} \quad . \tag{IBP}$$

a) Reduce (IBP) to a problem with homogeneous boundary conditions (and possibly, a nonzero forcing term).

b) Use the eigenfunction expansion method to construct a series solution for the resulting series solution for the problem in part a) (and thus for (IBP)). **Explicitly calculate all the coefficients in the series representing the solution**.