FALL 2002 ODE/PDE PRELIMINARY EXAM

DO 3 PROBLEMS FROM PART I AND 3 PROBLEMS FROM PART II. YOU MUST CLEARLY INDICATE WHICH 6 PROBLEMS ARE TO BE GRADED.

PART I: ODE

1. Work parts (a) and (b).

   (a) Let $\Omega \subset D$ be a compact positively invariant set for $\dot{x} = f(x)$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function with $\dot{V} \leq 0$ in Ω. Let M be the largest invariant set in $E = \{x \in \Omega : \dot{V}(x) = 0\}$. For this setting, give a precise statement of LaSalle's Invariance Theorem.

   (b) Consider the system $\left\{ \begin{array}{l} \dot{x} = y^3 - xy^2 \\ \dot{y} = -x + x^2 y \end{array} \right\}$. Apply the LaSalle Invariance Theorem to show that the origin is asymptotically stable and give an estimate for the region of attraction. (Hint: Consider $V(x,y) = \alpha x^2 + \beta y^4$.)

2. For $n \times n$ matrices A and B, prove that $e^{(A+B)t} = e^{At}e^{Bt}$ for all t if and only if $AB = BA$.

3. Use the transformations, $v(x) = (1+x)^{1/2}u(x)$ and $y = \ln(1+x)$ to completely solve the eigenvalue problem,

$$\frac{d}{dx}\left[(1+x)^2 \frac{du}{dx}\right] + \lambda u = 0, \quad 0 < x < 1,$$
$$u(0) = 0,$$
$$u(1) = 0.$$

4. Do both parts (a) and (b).

   (a) Let $f : [-a, a] \to \mathbb{R}$, with $a > 0$, be a continuous function satisfying

$$(*) \quad \left\{ \begin{array}{ll} f(x) > 0, & 0 < x < a \\ f(x) = 0, & x = 0 \\ f(x) < 0, & -a < x < 0 \end{array} \right. .$$

   Show that
$$H(x,y) = \frac{1}{2}y^2 + \int_0^x f(\xi)\, d\xi$$
   is positive definite on $D = \{(x,y) : -a < x < a, \ -\infty < x < \infty\}$.

   (b) For $\ddot{x} + f(x) = 0$, where f satisfies $(*)$, show that $(x, \dot{x}) = (0,0)$ is a stable critical point in the phase space $D \subset \mathbb{R}^2$ (here $y = \dot{x}$).

5. Find the Green's function for,

$$y'' - \gamma^2 y = 0,$$
$$y'(0) = 0,$$
$$y(1) = 0,$$

   where γ is a positive constant.

PART II: PDE

1. Prove that the only bounded harmonic functions on $\mathbb{R}^n$ are constants.

2. Solve the initial value problem,

$$u_{tt} - u_{xx} = 1,\ t > 0,\ x \in \mathbb{R}$$

$$u(x,0) = \begin{cases} 1, & -1 \le x \le 1, \\ 0, & |x| > 1, \end{cases}$$

$$u_t(x,0) = 0.$$

3. Find the solution $u(x,y,z)$ of the equation,

$$xu_x + yu_y + u_z = u$$

with Cauchy data, $u(x,y,0) = \phi(x,y)$ where ϕ is a given C^1 function.

4. Find the solution of the initial-boundary value problem,

$$u_t - u_{xx} - 2u_x = 0, \quad t > 0,\ 0 < x < 1,$$

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = e^{-x}\sin(11\pi x).$$

5. Let Ω be an open subset of $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$. Let u_0 and u_1 be given C^2 functions on Ω. Show the problem,

$$u_{tt}(x,t) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}, \quad x = (x_1, x_2, \ldots, x_n) \in \Omega, \quad t > 0$$

$$u(x,0) = u_0(x), \quad x \in \Omega,$$

$$u_t(x,0) = u_1(x), \quad x \in \Omega,$$

$$u(x,t) = 0, x \in \partial\Omega,\ t > 0,$$

has at most one solution $u \in C^2(\overline{\Omega} \times [0,\infty))$.