SPRING 2002 ODE/PDE PRELIMINARY EXAM

Do 3 problems from Part I and 3 problems from Part II. You must clearly indicate which 6 problems are to be graded.

PART I: ODE

1. Let $A_{n\times n}$ be a real matrix and let $f:\mathbb{R}\to\mathbb{R}^n$ be a continuous function which is periodic with period T. Consider the differential equation,

$$\frac{dx(t)}{dt} \;=\; Ax(t) \;+\; f(t). \tag{1}$$

   a) Under the assumption that all eigenvalues of A have nonzero real part, show there exists a unique initial condition x_0 which produces a (unique) T-periodic solution $x^*(t)$ to the equation (1).

   b) If, in addition, all eigenvalues of A have strictly negative real parts, show that the periodic solution x^* is stable, i.e., if x is an arbitrary solution of (1) then $\big(x(t)-x^*(t)\big)\xrightarrow{t\to\infty}0$.

2. For each of the following systems determine whether the origin is stable, asymptotically stable, unstable or totally unstable. State any theorems referred to in each case.

   a) $$\begin{cases}\dot{x}_1 = x_2 - x_1^5\\ \dot{x}_2 = -x_1 - x_1^3 - x_2^7\end{cases}$$

   b) $$\begin{cases}\dot{x}_1 = x_2 + x_1^5\\ \dot{x}_2 = -x_1 - x_1^3 + x_2^7\end{cases}$$

3. Let $y:\mathbb{R}\to\mathbb{R}$ be a nonzero solution of the differential equation,

$$8\frac{d^2y(x)}{dx^2} \;-\; 8\frac{dy(x)}{dx} \;+\; e^x y(x) = 0.$$

   a) Show that y has infinitely many zeros in the interval $(0,\infty)$ and at most one zero in the interval $(-\infty,0)$.

   b) Label the positive zeroes as $x_1 < x_2 < \cdots < x_n < \cdots$. Show that $(x_{n+1}-x_n)\xrightarrow{n\to\infty}0$.

4. Prove the following:
   Theorem: Suppose $f:\mathbb{R}\to\mathbb{R}$ is a globally Lipschitz continuous function. Then, for each $x_0\in\mathbb{R}$ there exists a unique C^1 function $x:\mathbb{R}\to\mathbb{R}$ (i.e., defined for all $t\in\mathbb{R}$) such that $x(0)=x_0$ and

$$\frac{dx(t)}{dt} \;=\; f(x(t)), \qquad \text{for all } t\in\mathbb{R}.$$

5. Consider the linear time varying system,

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1+t & \sin(t) & e^{-t}\cos(t)\\ -e^t & -t+2 & -e^t\\ e^t\sin(t) & e^t\sin(2t) & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

   Let $\Phi(t)$ denote a fundamental matrix of solutions.

   a) Derive from first principles a differential equation satisfied by $\det(\Phi(t))$.

   b) Use (a) to show that the system is unstable. You must carefully state the definition of type of stability applicable here.

PART II: PDE

1. Solve the partial differential equation

$$uu_x + u_t = x; \quad x \in \mathbb{R},$$

$$\text{subject to} \quad u(x,0) = 0.$$

2. Solve the Initial Value Problem in $\mathbb{R}^3$ for the wave equation

$$u_{tt} = \triangle u \ , \ x \in \mathbb{R}^3, \ t \geq 0$$

subject to initial conditions $u(x,0) = 0$ and $u_t(x,0) = x_3^3$.

3. Consider the eigenvalue problem

$$\frac{d^2\varphi(x)}{dx^2} \ + \ 3x^2\frac{d\varphi(x)}{dx} + \lambda\varphi(x) = 0$$

with boundary conditions $\varphi(1) = 0$ and $\varphi(2) = 0$.

a) Carefully argue that there are infinitely many eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ and corresponding eigenfunctions $\{\varphi_n\}_{n=1}^{\infty}$ of the problem. State an orthogonality relationship satisfied by the eigenfunctions.

b) Use the results from part (a) to derive a formal solution in the form of a "generalized" Fourier series involving φ_n for the partial differential equation

$$\begin{aligned} u_{tt} &= u_{xx} + 3x^2u_x \ , \quad x \in [1,2], \quad t > 0 \\ u(1,t) &= u(2,t) = 0 \ \text{ for all } t \geq 0, \\ u(x,0) &= 0, \quad u_t(x,0) = g(x) \ \text{ for all } x \in [1,2]. \end{aligned}$$

You must derive formulae that will enable computation of coefficients in the Fourier series. You may assume without proof that the λ_n are strictly positive.

c) Write out the explicit solution of the problem in part (b) when $g(x) = 3\varphi_2(x) + 4\varphi_6(x)$ where φ_n are the eigenfunctions given in part (a).

4. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function.

a) State the Mean Value Theorem for harmonic functions.

b) Use the Mean Value Theorem to show that

$$|u(x)|^2 \leq \frac{1}{\pi R^2} \iint\limits_{\|y-x\|_2 \leq R} u^2(y)\, dy_1 dy_2$$

for all $x \in \mathbb{R}^2$ and all $R > 0$. Here $\|\cdot\|_2$ denotes the Euclidian norm.

b) If u satisfies $\displaystyle\iint\limits_{\|y\|_2 \leq R} u^2(y)\, dy_1 dy_2 \leq \sqrt{R}$ for all $R > 0$, prove that $u(x) = 0$ for all $x \in \mathbb{R}^2$.

5. Let $z = z(x,t)$ denote a solution of the heat equation $z_t(x,t) - z_{xx}(x,t) = 0$ in the region $Q = \{(x,t) : 0 < x < \ell, \ \ 0 < t \leq T\}$ (with $\ell > 0$ and $T > 0$) which is continuous in the closed region $\overline{Q}$. Prove that the maximum of z is achieved on the initial line $\mathcal{S}_b = \{(x,0) : 0 < x < \ell\}$ or on the boundary lines $\mathcal{S}_0 = \{(0,t) : 0 < t \leq T\}$, $\mathcal{S}_\ell = \{(\ell,t) : 0 < t \leq T\}$.