FALL 2003 ODE/PDE PRELIMINARY EXAM

Do 3 problems from Part I and 3 problems from Part II. You must clearly indicate which 6 problems are to be graded.

PART I: ODE

1. Consider the linear system of ordinary differential equations of the form

$$\frac{dx}{dt}(t) = A(t)x(t) \quad x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^{n+m}, \quad \text{where } A = \left[\begin{array}{c|c} A_{11} & A_{12}(t) \\ \hline 0 & A_{22} \end{array}\right] \qquad (*_1)$$

and A_{11}, A_{22} are $n \times n$ matrices (constant coefficient) and $A_{12}(\cdot)$ is an $n \times n$ matrix whose entries are continuous functions of t.

   (a) Write down the general solution of $(*_1)$ in terms of $e^{A_{11}t}$, $e^{A_{22}t}$, $A_{12}(\cdot)$ and arbitrary constant vectors.

   (b) Suppose there are constants $c > 0$, $\lambda > 0$ such that $\|e^{A_{11}t}\| \le ce^{-\lambda t}$, $\|e^{A_{22}t}\| \le ce^{-\lambda t}$ and $\|A_{12}(t)\| < c$ for all $t \ge \tau \ge 0$. Show that $x(t) \xrightarrow{t\to\infty} 0$

2. Consider the linear system, $\dot{x}(t) = Ax(t)$, where $x(t) \in \mathbb{R}^n$, and A is an $n \times n$ symmetric matrix. Let $m < n$ denote a fixed integer such

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m > \lambda_{m+1} \ge \cdots \ge \lambda_n$$

denote the eigenvalues of A. Let V denote the sum of the eigenspaces of $\{\lambda_j\}_{j=1}^m$.
Let $\pi : \mathbb{R}^n \to V$ denote the orthogonal projection of $\mathbb{R}^n$ onto V. Finally let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$.

   (a) Show that $\pi e^{At} = e^{At}\pi$ for all $t \in \mathbb{R}$.

   (b) Show that $\|e^{At}x - e^{At}\pi(x)\| \le e^{\lambda_{m+1}t}\|x - \pi(x)\|$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

   (c) Consider the case $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Show that

$$\|e^{At}x - e^t x_1 e_1\| \le e^{-t}\sqrt{x_2^2 + x_3^2} \quad \text{for all} \quad t \in \mathbb{R}.$$

3. Consider the boundary value problem $\boxed{y''(x) = f(x) \quad y(0) = \alpha, \ \ y(1) = \beta \qquad (**_1)}$ where f is a continuous function on $[0,1]$.

   (a) Find a function $v(x)$ so that the $\widetilde{y}(x) = y(x) - v(x)$ satisfies the transformed homogeneous boundary value problem $\boxed{\widetilde{y}''(x) = f(x) \quad \widetilde{y}(0) = 0, \ \ \widetilde{y}(1) = 0 \qquad (**_2)}$

   (b) Find the Green's function for the problem $(**_2)$ and use it to provide a formula for the solution to the problem $(**_1)$.

4. Consider $\dot{x}(t) = f(x(t))$, where $x(t) \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^2 and satisfies $f(0) = 0$, and $\|\nabla f(x)\| < L$ for all $x \in \mathbb{R}^n$ and some $L > 0$. Let $t \mapsto x(t) : [0, a) \to \mathbb{R}^n$ be a solution.

   (a) Show that $\|x(t)\| \le e^{Lt}\|x(0)\|$ for all $t \in [0, a)$.

   (b) Hence argue that the solution exists for all $t > 0$. HINT: You may use Gronwall's inequality, *If* $\phi : [0, a) \to [0, \infty)$ *such that* $\phi(t) \le K + \int_0^t L\phi(s)\,ds \ \ \forall \ \ t \in [0, a)$, *then* $\phi(t) \le Ke^{Lt}$. *Note that this inequality is valid only when* K, L *and* ϕ *are nonnegative.*

PART II: PDE

1. Solve the first order linear initial value problem $u_x + u_t + tu = 0, \quad x \in \mathbb{R},\ t > 0$ satisfying $u(x, 0) = x^2, \quad x \in \mathbb{R}$.

2. Given that $u(x)$ is harmonic on

$$\left\{x \,:\, \frac{1}{2} \le \|x\| \le 1\right\} \subset \mathbb{R}^2.$$

   Let

$$M_1 = \max\{u(x) \,:\, \|x\| = 1\} \ \text{ and } \ M_2 = \max\left\{u(x) \,:\, \|x\| = \frac{1}{2}\right\}.$$

   Assume that $M_2 < M_1$ and $u(1, 0) = M_1$. Show that

$$\left.\frac{\partial}{\partial x_1}u(x)\right|_{x=(1,0)} \ge \frac{(M_1 - M_2)}{\ln(2)}.$$

   (HINT: Consider $v_\epsilon(x) = u(x) - \epsilon \ln(\|x\|)$ and use the maximum principle.)

3. (a) Write down the Poisson formula for a harmonic function u on $B(0, R) \subset \mathbb{R}^n$.

   (b) Suppose u is harmonic on $\overline{B(0, R)}$ and $u \ge 0$. Use part (a) to show that

$$\frac{R^{n-2}(R - \|\xi\|)}{(R + \|\xi\|)^{n-1}}u(0) \le u(\xi) \le \frac{R^{n-2}(R + \|\xi\|)}{(R - \|\xi\|)^{n-1}}u(0), \qquad \text{for all } \ \|\xi\| < R.$$

4. Consider the equation on $\mathbb{R}^2$,

$$x^2 u_{xx} + 2xu_{xy} + u_{yy} = u_y. \qquad (*_2)$$

   (a) Show that the equation is parabolic.

   (b) Find the characteristics, and show that $\alpha(x, y) = xe^{-y}$ and $\beta(x, y) = y$, is a canonical transformation.

   (c) Show that, in the transformed coordinates, the equation becomes $u_{\beta\beta} - u_\beta = 0$ and hence find the general solution of $(*_2)$.

5. Consider the initial value problem for the wave equation in $\mathbb{R}^3$

$$u_{tt} = \Delta_x u, \ \ x \in \mathbb{R}^3, \quad t \ge 0,$$

$$u(x, 0) = g(x), \quad u_t(x, 0) = h(x), \quad \forall \ \ x \in \mathbb{R}^3.$$

   (a) Write down the Kirchoff's formula for the solution.

   (b) Solve the equation explicitly when $g(x) = \|x\|^2$ and $h(x) = x_1x_2$.