SPRING 2003 ODE/PDE PRELIMINARY EXAM

Do 3 problems from Part I and 3 problems from Part II. You must clearly indicate which 6 problems are to be graded.

PART I: ODE

1. Construct an example, $\dot{x}(t) = A(t)x(t)$, $x(t) \in \mathbb{R}^2$, where $A : \mathbb{R} \to \mathbb{R}^{2\times 2}$ is a matrix valued function, such that the eigenvalues of $A(t)$ have negative real part for each $t \in \mathbb{R}$, but the origin is unstable.

2. a) Show that for a second order equation of the form $\ddot{y}(t) + a(t)y(t) = 0$, the Wronskian of any two solutions is a constant.

   b) The third order equation $\dddot{y} + 3t^{-1}\ddot{y} - 2t^{-2}\dot{y} + 2t^{-3}y = 0$ for $t > 0$ has a fundamental system of solutions given by $y_1(t) = t$, $y_2(t) = t\log(t)$, $y_3(t) = 1/t^2$. Using an initial time $t_0 = 1$ show that the Wronskian $W(t) = 9/t^3$ for $t \geq 0$.

   c) Give an example of functions $y_1(t)$ and $y_2(t)$ which are linearly independent on $\mathbb{R}$ and yet the Wronskian $W(t) \equiv 0$ for all $t \in \mathbb{R}$.

3. For the system $\left\{ \begin{array}{l} \dot{x} = y - x^3 + xy^3 \\ \dot{y} = -x - y^5 \end{array} \right.$, show that the origin is a globally asymptotically stable equilibrium. (Hint: Consider $V(x,y) = 1/2(x^2 + y^2)$ and use Young's inequality).

4. Consider the Sturm-Liouville eigenvalue problem

$$\frac{d}{dx}\left(k(x)\frac{dy(x)}{dx}\right) + \lambda y(x) = 0, \quad y(0) = 0, y(1) = 0,$$

   where $k \in C^2([0,1])$ and $k(x) > 0$ for all $x \in [0,1]$.

   You may assume without proof that all eigenvalues are real, there are infinitely many eigenvalues which can be ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, having no finite cluster points. Let v_n be the eigenfunction associated with λ_n, $n = 1, 2, \cdots$.

   a) Show that $\lambda_1 > 0$.

   b) Show that y_n has exactly $(n-1)$ zeros in $(0,1)$.

   c) Let k_1 and k_2 be positive constants such that $k_1 < k(x) < k_2$ for all $x \in [0,1]$. Show that

$$\frac{k_1^2(n+1)^2\pi^2}{k_2} < \lambda_n < \frac{k_2^2(n+1)^2\pi^2}{k_1}.$$

5. a) Let $\alpha \in C^1[0,\infty)$ satisfy $\dot{\alpha}(t) \leq \alpha(t)$, $\forall\, t \geq 0$ and $\alpha(0) = 1$. Use Gronwall's inequality (or whatever method you prefer) to show that $\alpha(t) \leq e^t$.

   b) Suppose $x \in C^1[0,\infty)$ satisfies $\dot{x}(t) = -2x(t) + e^{-t}\alpha(t)$, where $\alpha(\cdot) \geq 0$ is as in part a). Show that $x(t)$ is bounded.

PART II: PDE

1. Solve the quasilinear Cauchy problem

$$xu_x + yuu_y = -xy$$

subject to $u = 5$ on $xy = 1$ and $x > 0$.
(Hint: It might be useful to compute $\frac{d}{dr}(x(r,s)y(r,s))$)

2. Solve the initial value problem,

$$u_{tt} = u_{xx} + u_{yy},\ t \geq 0,\ (x,y) \in \mathbb{R}^2$$

$$u(x,y,0) = x, \quad u_t(x,y,0) = y, \quad \forall\ (x,y) \in \mathbb{R}^2.$$

3. Consider the equation,

$$u_{tt} = \Delta u - u,\ t \geq 0,\ (x,y) \in \mathbb{R}^2$$

$$u(x,y,0) = \varphi(x,y), \quad u_t(x,y,0) = \psi(x,t), \quad \forall\ (x,y) \in \mathbb{R}^2,$$

where $\varphi, \psi \in C^2(\mathbb{R}^2)$ are given and Δ is the Laplacian. Obtain an energy inequality and use it to show that C^2 solutions of the problem are unique. (You do not have to prove existence).

4. For $x \in \mathbb{R}^n$ let $\|x\|$ denote the Euclidean norm of x. Work both parts a) and b):

   a) Show that $v(x) = \|x\|^{2-n}$ is harmonic on $\mathbb{R}^n \backslash \{0\}$ if $n \geq 3$.

   b) Let $\Omega = B_1(0)$ (the open unit ball) in $\mathbb{R}^4$, and suppose that $u \in C^2(\overline{\Omega})$ (where $\overline{\Omega}$ is the closure of Ω) is such that $\Delta u \geq 0$ on Ω. Suppose u takes its maximum value at $x_0 \in \partial\Omega$ (the boundary of Ω). Let $\nu = x_0/\|x_0\|$. Show that either $\frac{\partial u}{\partial \nu}(x_0) > 0$ or u is a constant on Ω.
   (Hint: let $w(x) = u(x) - \epsilon(1 - \|x\|^{-2})$ on an appropriate subdomain of Ω with small ϵ and use the maximum principle.)

5. **Setup:** Let $\Omega \underset{\text{open}}{\subset} \mathbb{R}^n$. A function $v : \Omega \to \mathbb{R}$ is said to be pseudo-subharmonic on Ω if

$$v(\xi) \leq \frac{1}{\omega_n} \int\limits_{\|x\|=1} v(\xi + rx)\, dS_x \quad \text{for every} \quad \xi \in \Omega$$

provided that $r > 0$ is small enough. Here ω_n denotes the measure of the unit sphere in $\mathbb{R}^n$ and dS_x denotes surface measure on $\|x\| = 1$.

Problem: Let $u_1, \ldots, u_k \in C(\overline{\Omega})$ be pseudo-subharmonic in Ω and define

$$u(x) = \max\{u_1(x), \ldots, u_k(x)\} \quad \text{for} \quad x \in \overline{\Omega}.$$

Show that u is pseudo-subharmonic on Ω.