FALL 2004 ODE/PDE PRELIMINARY EXAM

DO 3 PROBLEMS FROM PART I AND 3 PROBLEMS FROM PART II. YOU MUST CLEARLY INDICATE WHICH 6 PROBLEMS ARE TO BE GRADED.

PART I: ODE

1. Consider the first order nonlinear system of differential equations for two unknown functions $x_1 = x_1(t)$ and $x_2 = x_2(t)$, $t \in \mathbb{R}$, (Note: $\dot{x} = \dfrac{dx}{dt}$).

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \sin(x_1^2 - 2x_1x_2) + 2 \\ \dot{x}_2 &= x_1\sqrt{1+\cos^2(x_1+x_2)} + 5x_2\exp(-x_1^2). \end{aligned}}$$

   (a) Write this system in the form $\boxed{\dfrac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A(x)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \boldsymbol{b}}$, where $\boldsymbol{b}$ is a column vector in $\mathbb{R}^2$, and $\|A(\cdot)\|$ is bounded (Here $\|\cdot\|$ denotes any operator norm).

   (b) Using Gronwall's inequality or otherwise, show that all solutions exist on the entire time interval $(-\infty, \infty)$. You must carefully state any theorems used in your proof.

2. Given the Boundary Value Problem $\begin{cases} y''(x) + y(x) = -f(x), \quad 0 \le x \le \ell \\ y(0) = 0, \ y(\ell) = 0 \end{cases}$

   (a) Find values of ℓ for which a Green's function does not exist. Pick any such ℓ and give an example of a function f for which the corresponding problem has a solution. Is the solution unique?

   (b) Pick any value of ℓ for which the Green's function exists, construct the Green's function, and give a formula for the solution of the boundary value problem (here the formula will involve a general function f).

3. Consider the system of ordinary differential equations for the unknown functions $x_j = x_j(t)$, $j = 1, 2, 3$, $t \in \mathbb{R}$.

$$\boxed{\begin{cases} \dot{x}_1 &= -x_2x_3 \\ \dot{x}_2 &= x_1x_3 \\ \dot{x}_3 &= -\frac{1}{3}x_1x_2 \end{cases}}$$

   Define $H_1(x) = x_1^2 + 2x_2^2 + 3x_3^2$ and $H_2(x) = x_1^2 + 4x_2^2 + 9x_3^2$.

   (a) Show that $H_1(x)$ and $H_2(x)$ remain constant along any solution of the system.

   (b) Consider the solution with initial condition $x(0) = (1, 1, 0)$. Use part (a) to show that the resulting solution satisfies

$$x_1^2(t) - 3x_3^2(t) = 1 \quad \text{and} \quad x_2^2(t) - 3x_3^2(t) = 1.$$

4. Consider the linear systems (LH) and (LNH) where $x(t), f(t) \in \mathbb{R}^n$, $A(t)$ is a real $n \times n$ matrix, and $A(t)$, $f(t)$ are continuous on $\mathbb{R}$.

$$\boxed{\begin{aligned} \dot{x}(t) &= A(t)x(t), && (LH) \\ \dot{x}(t) &= A(t)x(t) + f(t), && (LNH) \end{aligned}}$$

   (a) Prove that the set of all solutions of (LH) forms an n-dimensional vector space.

   (b) Define what is meant by a fundamental matrix of (LH) and explain why it exists.

   (c) Assume that a fundamental matrix is known and derive the variation of parameters formula for the solution of (LNH) subject to the initial condition $x(t_0) = x_0$.

PART II: PDE

1. Consider the quasilinear differential equation

$$u(x,y)u_x(x,y) + u_y(x,y) = 1, \quad x, y \in \mathbb{R},$$

subject to the initial condition

$$u(x,x) = 0, \quad x \in \mathbb{R}.$$

Solve for $u(x,y)$.

2. Find the explicit solution $u(x,t)$ of the nonhomogeneous IVP

$$\begin{aligned} &u_{tt}(x,t) = u_{xx}(x,t) + x, \quad x \in \mathbb{R}, \quad t > 0, \\ &u(x,0) = x^2, \quad u_t(x,0) = 0. \end{aligned}$$

3. (a) Consider the Dirichlet Problem for

$$\Delta u = 0, \quad x \in \Omega; \quad u(x) = f(x), \quad x \in \partial\Omega, \tag{DP}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$. Let $f_1, f_2 \in C^2(\partial\Omega)$ and let $u_1, u_2 \in C^2(\overline{\Omega})$ be solutions of (DP) corresponding to f_1 and f_2, respectively. Prove that for any $\epsilon > 0$, if

$$|f_1(x) - f_2(x)| \leq \epsilon, \quad \text{for all} \quad x \in \partial\Omega,$$

then

$$|u_1(x) - u_2(x)| \leq \epsilon, \quad \text{for all} \quad x \in \overline{\Omega}.$$

State carefully any theorems that are used in the proof.

(b) Determine whether the Dirichlet Problem in $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : y > 0\}$

$$\Delta u(x,y) = 0, \quad (x,y) \in \mathbb{R}^2_+, \quad u(x,0) = x, \quad x \in \mathbb{R} \tag{DP}$$

has a unique solution. Prove or give a counterexample.

4. Consider the following initial boundary value problem for the heat equation with nonhomogeneous boundary conditions:

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \quad x \in (0,1), \quad t > 0, \\ &u(0,t) = 0, \quad u_x(1,t) = t \qquad . \\ &u(x,0) = x. \end{aligned} \tag{$*$}$$

(a) Reduce $(*)$ to a problem with homogeneous boundary conditions (and possibly, a nonzero forcing term).

(b) Use separation of variables to construct a series solution for the problem in part (a). **You must explicitly calculate all the coefficients in the series representing the solution**.

(c) Use the solution obtained in part (b) to obtain a solution to $(*)$.