SPRING 2004 ODE/PDE PRELIMINARY EXAM

DO 3 PROBLEMS FROM PART I AND 3 PROBLEMS FROM PART II. YOU MUST CLEARLY INDICATE WHICH 6 PROBLEMS ARE TO BE GRADED.

PART I: ODE

1. Consider the Lorenz system $\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = xy - \beta z \end{cases}$, with $\rho > 0$, $\beta > 0$ and $\sigma \in (0, 1)$.

   Show that

   the origin is globally asymptotically stable.

   (**hint**: Use the Lyapunov function $V(x, y, z) = \rho x^2 + \sigma y^2 + \sigma z^2$.)

2. Consider a linear system $\dot{x}(t) = A(t)x(t)$, $x(t) \in \mathbb{R}^5$ and fundamental matrix $\Phi(t)$ with

$$\Phi(t) = \left[\begin{array}{cc|cc|c} \cos(t) & -\sin(t) & 0 & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 & 0 \\ \hline 0 & 0 & \dfrac{(4e^t - e^{-2t})}{3} & \dfrac{-2(e^t - e^{-2t})}{3} & 0 \\ 0 & 0 & \dfrac{2(e^t - e^{-2t})}{3} & \dfrac{(4e^{-2t} - e^t)}{3} & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

   (a) Compute $A(t)$. (**hint**: $A(t)$ is actually a constant matrix.)
   (b) Compute all periodic orbits and equilibria.
   (c) Determine the stability of the system.

3. Consider the ordinary differential equation $\dot{x}(t) = F(x(t), t)$, where $x(t) \in \mathbb{R}^n$, and $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\|F(x, t)\| \le L(\|x\| + t), \quad \text{for all} \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}$$

   where L is a positive constant.

   (a) Prove the following Gronwall's Inequality: Let $f_1(t)$, $f_2(t)$, $p(t)$ be continuous on $[a, b]$ and $p \ge 0$. If $f_1(t) \le f_2(t) + \int_a^t p(s) f_1(s)\, ds$, $t \in [a, b]$, then

$$f_1(t) \le f_2(t) + \int_a^t p(s) f_2(s) \exp\left(\int_s^t p(u) du \right) ds.$$

   (b) Obtain an explicit bound for $\|x(t)\|$ in terms of L and t for all $t \ge 0$.

4. (a) For functions u and v in $C^4[0, 1]$ that satisfy

$$u(0) = v(0) = 0, \; u'(0) = v'(0) = 0, \; u(1) = v(1) = 0, \; u''(1) = v''(1) = 0,$$

   show that $\displaystyle\int_0^1 \left(u\frac{d^4 v}{dx^4} - v\frac{d^4 u}{dx^4} \right) dx = 0.$

(b) Show that the eigenfunctions corresponding to different eigenvalues of the problem

$$\frac{d^4\varphi(x)}{dx^4} + \lambda e^x \varphi(x) = 0, \quad x \in [0,1], \quad \varphi(0) = \varphi'(0) = \varphi(1) = \varphi''(1) = 0$$

are orthogonal with respect to an appropriate weight function. What is the weight function?

PART II: PDE

1. Transform the equation

$$x^2 u_{xx} - y^2 u_{yy} + 3x u_x - y u_y = 0, \quad x > 0, \ y > 0,$$

into a canonical form.

2. Consider the partial differential equation

$$u_{tt} + \Delta_x u + \varphi(x) u_t = 0, \quad x \in \mathbb{R}^n, \quad t \geq 0, \quad \text{where } \varphi(x) \geq 0 \ \text{ for all } \ x \in \mathbb{R}^n.$$

Suppose that $u \in C^2(\mathbb{R}^n \times [0,\infty))$ is a solution. Show that

$$\int_{B_{(R-T)}(x_0)} \left[u_t^2(x,T) + \|\nabla_x u(x,T)\|_2^2\right] dx \leq \int_{B_R(x_0)} \left[u_t^2(x,0) + \|\nabla_x u(x,0)\|_2^2\right] dx,$$

for all $x_0 \in \mathbb{R}^n$, all $R > 0$ and for all T satisfying $0 < T < R$.

3. Given that $u(x)$ is harmonic on

$$\left\{x = (x_1, x_2) \ : \ \frac{1}{2} \leq \|x\| \leq 1\right\} \subset \mathbb{R}^2,$$

let

$$M_1 = \max\{u(x) \ : \ \|x\| = 1\} \ \text{ and } \ M_2 = \max\left\{u(x) \ : \ \|x\| = \frac{1}{2}\right\}.$$

Assume that $M_2 < M_1$ and $u(1,0) = M_1$. Show that

$$\left.\frac{\partial}{\partial x_1} u(x)\right|_{x=(1,0)} \geq \frac{(M_1 - M_2)}{\ln(2)}.$$

(**hint**: Consider $v_\epsilon(x) = u(x) - \epsilon \ln(\|x\|)$ and use the maximum principle.)

4. Solve

$$u_{tt}(x,t) - u_{xx}(x,t) = t\sin(\pi x), \ x \in [0,1], \quad t \geq 0,$$

$$u(x,0) = \sin(3\pi x), \quad u_t(x,0) = \sin(5\pi x).$$