FALL 2005 ODE/PDE PRELIMINARY EXAM

DO 3 PROBLEMS FROM PART I AND 3 PROBLEMS FROM PART II. YOU MUST CLEARLY INDICATE WHICH 6 PROBLEMS ARE TO BE GRADED.

PART I: ODE

1. Given $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

   a) Discuss stability for the system $\dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x}$.

   b) Find a fundamental matrix for the system in a).

   c) Solve the problem $\dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}(t)$ with $\boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{b}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 1 \end{bmatrix}$.

2. For the system of equations $\begin{cases} x' = 2x - xy \\ y' = -y + xy \end{cases}$

   a) Find all equilibria.

   b) Prove that if $x(0) > 0$ and $y(0) > 0$ then $x(t) > 0$ and $y(t) > 0$ for all $t \geq 0$.

   c) Show that all solutions with $x(0) > 0$ and $y(0) > 0$ are bounded and, in fact, periodic when $(x(0), y(0)) \neq (1, 2)$ (Hint: Use $E(x, y) = x - \ln(x) + y - 2\ln(y)$).

   d) Determine stability properties of the equilibria in part a).

3. Suppose that $\Phi(t)$ is a fundamental matrix for a linear system

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x}.$$

   Suppose there exists $M > 0$ such that $|\Phi(t)| \leq M$ for all $t \geq 0$. Show that $\boldsymbol{x} = 0$ is stable.

4. Determine whether each of the following has a limit cycle or not:

   a) $\begin{cases} x' = x^2 + 2y^2 \\ y' = x - 2 \end{cases}$ b) $\begin{cases} x' = -12xy + x^3 \\ y' = 4y \end{cases}$

PART II: PDE

1. Consider the first order partial differential equation $u_t - e^{-u}u_x = 0$ $(*)$

   a) Solve the characteristic ODEs for $(*)$.

   b) Find an explicit nonconstant solution of $(*)$ in the form $u(x,t) = f(x/t)$. (Note: The solution you find may only be defined on a subdomain in the x, t plane.)

2. a) Find the Green's function for the boundary value problem (BVP)

$$y'' = 0, \ y(0) = 0, \quad y(1) - 2y'(1) = 0.$$

   b) Use your answer in part a) to solve the BVP $y'' = x, \ y(0) = 0, \quad y(1) - 2y'(1) = 0$.

3. Use Duhamel's principle to solve the initial value problem for the non-homogeneous wave equation

$$\begin{aligned} w_{tt} &= w_{xx} + \sin(x - t), \quad x \in \mathbb{R}, \quad t > 0, \\ w(x,0) &= 0, \quad x \in \mathbb{R}, \\ w_t(x,0) &= 0, \quad x \in \mathbb{R}. \end{aligned}$$

4. Consider the following modified heat equation

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t) - u(x,t), \quad 0 < x < 1, \quad t > 0, \\ &u(x,0) = f(x), \quad 0 < x < 1, \\ &u(0,t) = 1, \quad u(1,t) = 0, \quad 0 < t < T\ . \end{aligned} \tag{1}$$

   (a) Find the steady state solution $u(x,t) = u_{\text{ss}}(x)$.

   (b) Use an energy argument on the function

$$w(x,t) = u(x,t) - u_{\text{ss}}(x)$$

   to describe the behavior of u as $t \to \infty$.