SPRING 2005 ODE/PDE PRELIMINARY EXAM

DO 3 PROBLEMS FROM PART I AND 3 PROBLEMS FROM PART II. YOU MUST CLEARLY INDICATE WHICH 6 PROBLEMS ARE TO BE GRADED.

PART I: ODE

1. Let $\boldsymbol{A} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

   a) Find a fundamental matrix for $\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x}$.

   b) Discuss the stability of equilibrium.

   c) Use the variation of parameters formula and your answer in part a) to solve the initial value problem
   $$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

2. (a) Show that there do NOT exist functions $p(x)$, $q(x)$ continuous in a neighborhood of $x = 0$ such that the ODE
   $$y'' + p(x)y' + q(x)y = 0$$
   has solutions $y_1(x) = x$ and $y_2(x) = \sin(x)$.

   (b) State and prove Abel's theorem on the Wronskian of two solutions to the equation
   $$y'' + p(x)y' + q(x)y = 0.$$
   (Recall Abel's theorem gives an explicit formula for the Wronskian.)

3. (a) Give a precise definition of stability (in the sense of Lyapunov) for a solution of the system of ODEs $x' = f(x,t)$, $x \in \mathbb{R}^n$.

   (b) If $k > 0$ is a constant, is $x = 0$ stable or unstable for $x' = xk^2 - x^3$?

   (c) In the system $\begin{cases} x' = xk^2 - x^3 \\ k' = 0 \end{cases}$ in which k is now a variable, is $x = 0$, $k = 0$ a stable equilibrium? Explain your reasoning carefully.

4. Consider the ODE, $\dot{x} = f(x,t)$, $x(t) \in \mathbb{R}$, $t \geq 0$. It is given that $f \in C^1$, $f(0,t) = 0$ and $f(\cdot,t)$ is periodic with respect to t with period T, i.e., $f(x, t+T) = f(x,t)$ for all x, t. Let $(t, x_0) \mapsto x(t; x_0)$ denote the solution at time t with initial state x_0 at $t = 0$. Suppose there exists $0 < \alpha < 1$ such that
   $$|x(T, x_0)| \leq \alpha |x_0| \quad \text{for} \quad |x_0| \leq 1$$
   (In particular, $x(t, x_0)$ exists for all $0 \leq t \leq T$ and $|x_0| \leq 1$.) Using Gronwall's inequality, or, otherwise show there exists $C > 0$, $\lambda > 0$ such that
   $$|x(t; x_0)| \leq Ce^{-\lambda t}|x_0| \quad \text{for all} \quad x_0 \in [-1, 1] \quad \text{and all} \quad t \geq 0.$$

PART II: PDE

1. Consider the quasilinear differential equation for $x, y \in \mathbb{R}$

$$xu_x(x,y) + yu_y(x,y) = u^2,$$

   a) Find the characteristics of this equation as curves in x and y.

   b) Find the form of the general solution to this equation.

   c) Find the solution to the initial value problem $u(1,y) = y^2$, $y \in \mathbb{R}$.

   d) Is the solution everywhere defined?

2. Find the Green's function for the problem

$$y'' - 3y' + 2y = f(x), \quad y(0) = 0, \quad y(1) = 0 \qquad (*)$$

   where f is a given continuous function on $[0,1]$. Use the Green's function to express the solution to $(*)$.

3. Use Duhamel's principle to find an explicit solution of

$$u_{tt}(x,t) = u_{xx}(x,t) + e^x, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x,0) = 0, \quad u_t(x,0) = 0.$$

4. a) Show that the characteristics of the wave equation $\frac{1}{c^2}u_{tt} - u_{xx} = 0$ are

$$\xi = x + ct \quad \text{and} \quad \eta = x - ct.$$

   b) Use the characteristics to transform the equation to a canonical form and thereby find the general form of the solution. Use this to express the solution to the initial value problem: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

   c) For the multidimensional wave equation $\frac{1}{c^2}u_{tt} - \Delta u = 0$ assume radial symmetry. Find the general solution in $\mathbb{R}^3$. (Hint: consider the transformation $w = ru$ where r is the radial coordinate.)

5. For any positive integer n consider the exterior boundary value problem:

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega = \{x \in \mathbb{R}^n : \|x\| > 1\} \\ u(x) &= 1, \quad \|x\| = 1, \end{aligned}$$

   a) Show that this problem has infinitely many solutions.

   b) Show that if for $n > 2$, $u(x) \to 0$ as $\|x\| \to \infty$, then the problem has a unique solution.