Spring 2006 ODE/PDE Preliminary Exam

Do 3 problems from Part I and 3 problems from Part II. You must clearly indicate which 6 problem are to be graded.

Part I.

Problem 1. Let $\Phi(t)$ be a real-valued continuous function on $[0,\infty)$. Consider the differential equation,

$$y''(t) - \cos^2(t)y' + \Phi(t)y(t) = 0, \quad t \geq 0.$$

Using Abel's formula or otherwise show that there exists an initial condition $(y(0), y'(0))$ such that the corresponding solution is unbounded.

Problem 2. Consider the system of ODE,

$$\frac{dx_1}{dt} = -x_2 + x_2x_3 - x_1^3$$
$$\frac{dx_2}{dt} = 2x_1 - x_2^3$$
$$\frac{dx_3}{dt} = -x_1x_2 - x_3^3$$

Show that the origin $(0,0,0)$ is a globally asymptotically stable equilibrium point.

Problem 3. Using Sturm's Theorem

A) Show that if a solution of the equation $y'' + (\cos x)y = 0$,

has at least two zeros in the interval $(\pi/2, 3\pi/2)$, then $y(x) \equiv 0$.

B) Suppose $k(x) \geq x$ for all {{{x in R}}}. Show that distances between consequent zeros of every nontrivial solution of the equation $y''+k(x)y = 0$ become arbitrarily small as $x \to \infty$.

Problem 4. Construct a system of ODE, for which the two functions,

$$\vec{y}_1 = e^{2x}\begin{pmatrix}1\\0\end{pmatrix}, \vec{y}_2 = e^{2x}\begin{pmatrix}x\\1\end{pmatrix}$$

will form a fundamental system of solutions.

Part II: PDE

Problem 1.

Using the Maximum principle show that if $u(x)$ is a solution of the Dirichlet problem

$$\Delta u(x) = f(x), \text{in } \{x \in R^3 : R_1 < |x| < R_2\}, where\ f(x) \geq 0$$
$$u\big|_{|x|=R_1} = M_1, u\big|_{|x|=R_2} = M_2, 0 \leq M_1 \leq M_2$$
,

then

$$u(x)\big| \leq \frac{M_2 - M_1}{1/R_2 - 1/R_1} r^{-1} + M_2 - \frac{M_2 - M_1}{1/R_2 - 1/R_1} R_2^{-1}.$$
$$when \quad |x| = r \quad for\ R_1 < r < R_2.$$

Problem 2.

Let $g(x) = \begin{cases} 1 - |x|, \text{ when } |x| \leq 1 \\ 0 \quad , \text{ when } |x| > 1 \end{cases}$. Sketch profiles of the solutions of two Cauchy problems at time $t = 3/4$.

$$u_{tt} - u_{xx} = 0 \qquad (1) \qquad\qquad u_t - u_{xx} = 0 \qquad (2)$$
$$u\big|_{t=0} = g(x), u_t\big|_{t=0} = 0 \qquad (1_0) \qquad\qquad u\big|_{t=0} = g(x). \qquad (2_0)$$

Explain major differences in the features of these two solutions.

Problem 3. Let D be the domain in R^2 shown in the figure 1

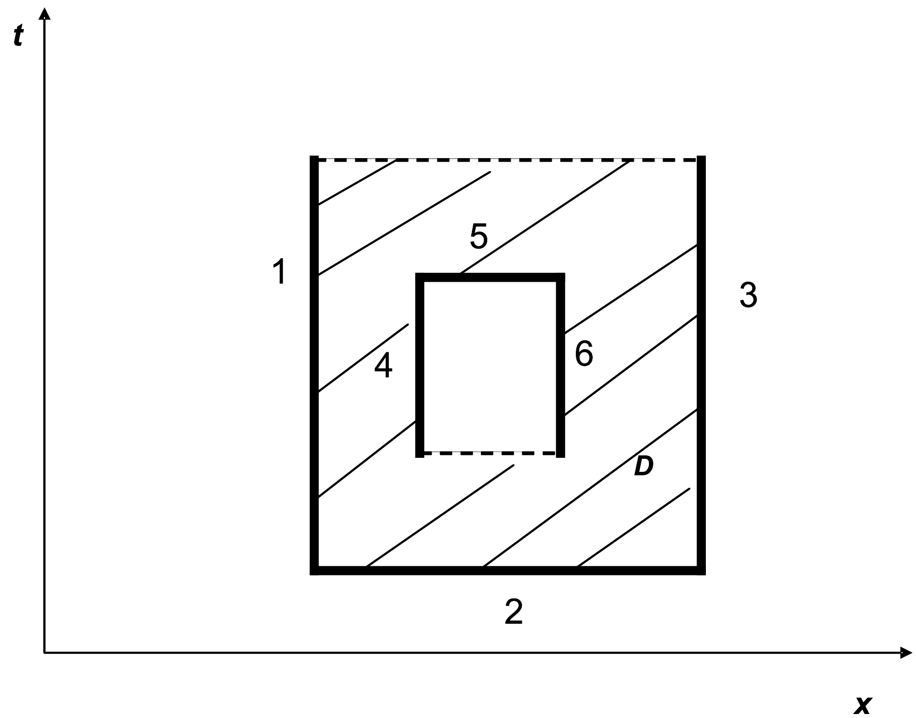

Figure 1: D is the shaded domain as shown above between two rectangles. Boundary conditions will be specified only on the darkened part Γ of the boundary, i.e. Γ consists of lines 1 though 6.

Let $u_1(x,t),\ and\ \ u_2(x,t)$ be two solutions of the problem

$$u_t - u_{xx} = f(x,t)$$
$$u|_\Gamma = g(x,t).$$
,

Let $f,$ and g are continuous functions. Prove that $u_1(x,t) = u_2(x,t)$, at each point of the domain D.

Problem 4. Derive an explicit formula for a function $u(x,t)$ which solves the initial value problem

$$\begin{cases} u_t + b \bullet \nabla u + c(t)u = f(x,t) & \text{in } R^n \times (0,\infty) \\ u|_{t=0} = g(x) \end{cases}$$

Here: $b \in R^n$ is a constant vector, $c = c_0 t^m$, c_0, m are constants, and f , g are continuous functions.