

## Spring 2007 ODE/PDE Preliminary Exam

Solve four problems from Part I and four problems from Part II. You must clearly indicate which eight problems are to be graded.

Part I.

Problem 1. Let  $u(x)$  and  $v(x)$  be real-valued nonnegative continuous functions such that

$$u'(x) \leq x^{-2} - u(x)v(x), \quad 0 < x_0 \leq x < \infty,$$

$$v'(x) \leq -v(x), \quad 0 < x_0 \leq x < \infty,$$

Show that  $u(x)$ , and  $v(x)$  are bounded.

Problem 2. Determine whether equilibrium  $x \equiv 0$ ,  $y \equiv 0$  is stable or unstable the following system:

$$x'(t) = y^3 + x^5$$

$$y'(t) = x^3 + y^5$$

Problem 3. Consider two-dimensional system:

$$\frac{dx}{dt} = y + x(1 - x^2 - y^2) \quad (1)$$

$$\frac{dy}{dt} = -x + y(1 - x^2 - y^2). \quad (2)$$

- i. Show that the set of points  $S = \{(x, y) \mid x^2 + y^2 = 1\}$  is an invariant set for the system. This means that if the initial point  $(x_0, y_0)$  is in this set  $S$ , then the solution trajectory remains in the set  $S$ . Hint: Work with the function  $f(x, y) = x^2 + y^2 - 1$ .
- ii. Show that for an initial condition in  $S$ , the solution is of the form  $x(t) = \cos(\omega t + \phi)$ ,  $y(t) = -\sin(\omega t + \phi)$ , where  $\omega > 0$  and  $\phi \in [0, 2\pi]$ . What is value of  $\omega$ ?
- iii. Obtain the equations of the system in polar coordinates. Using these equations, show that the set  $S$  is an  $\omega$  limit set - that is, every trajectory except the zero trajectory spirals towards this set as  $t \rightarrow \infty$ .

Problem 4. Prove Bendixson's criteria in (i) and use it solve the problem in (ii).

- i. Consider system

$$\frac{dx}{dt} = P(x, y) \quad (1)$$

$$\frac{dy}{dt} = Q(x, y), \quad (2)$$

where  $x, y, t \in \mathbb{R}$ . Let  $P(x, y)$  and  $Q(x, y)$  have continuous first partial derivatives in simply connected domain  $D \subset \mathbb{R}^2$  and assume that  $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$  is not identically zero and does not change sign in  $D$ . Then the given system has no periodic orbit in  $D$ . Hint: Use contradiction argument - assume a periodic orbit  $\Gamma$  with period  $T$  in  $D$  and use Green's theorem.)

- ii. Let  $\delta, \beta$  are continuous, and  $\alpha, \gamma$  are both continuous and positive in some simply connected domain  $D$ .

Show that the system:

$$\frac{dx}{dt} = \alpha(y)x + \beta(y) \quad (5)$$

$$\frac{dy}{dt} = \gamma(x)y + \delta(x), \quad (6)$$

has no non-constant periodic solution on that domain  $D$ .

Problem 5. Find all the solutions of the system:

$$xy_1'(x) = 2y_1 - y_2$$

$$xy_2'(x) = 2y_1 - y_2$$

Show that

- If the initial conditions  $y_1(x_0) = y_1^0$ ,  $y_2(x_0) = y_2^0$  are specified for  $x_0 \neq 0$ , the solution exists and is unique on the entire real axis; if  $x_0 = 0$ , the solution exists if  $2y_1^0 - y_2^0 = 0$  but is not unique.
- The Wronskian of linearly independent solutions equals  $cx$ , where  $c \neq 0$ .
- How can the fact that the Wronskian vanishes at only one single point be reconciled with the fact that it is not equal 0 identically?

Part II: PDE

Problem 1. Accept without proof  $u(x, y) = \frac{1-x^2-y^2}{y^2+1-x^2}$  is harmonic and positive for  $x^2 + y^2 < 1$ .

Since  $u(x, y) \equiv 0$ , for  $x^2 + y^2 = 1$  except at  $(1, 0)$ , is maximum principle valid for  $u(x, y)$ ? Explain.

Problem 2.

Let  $R_+^n = \{x : x \in R^n, x_1 \geq 0\}$ . Let function  $u \in C^2(R_+^n)$ , solves BVP in  $R_+^n$ .

$$\Delta u = 0 \quad \text{in} \quad \{x : x \in R^n, x_1 > 0\} \quad (1).$$

$$u(0, x_2, \dots, x_n) = 0 \quad (2).$$

Assume the existence of a positive constant  $M$  such that

$$u \leq M + o(x_1) \quad (3).$$

Prove that  $u \equiv 0$ .

(Hint: you may use Liouville's theorem).

Problem 3.

Let  $U \subset R^n$  be a bounded domain with a boundary allowing application of the Green formula. Let  $u(x, t)$  be a solution of IBVP

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U \times (0, \infty) \\ u|_{\partial U} = 0 \\ u(x, 0) = f(x) & \text{in } U \end{cases}$$

Where initial function  $f \in C(U)$ . Using energy method prove that  $\int_U (u(x, t))^2 dx \rightarrow 0$ , as  $t \rightarrow \infty$ . (Hint: you may use Poincare-Friedrich's inequality).

Problem 4.

a) Let  $x \in R^3$ , and let  $u(x, t)$  be a solution of the IVP

$$\begin{cases} u_t - \Delta u = 0 & \text{in } R^3 \times (0, \infty) \\ u(x, 0) = 0 \\ u_t(x, 0) = h(x) \end{cases}$$

Assume  $h(x)$  is a smooth function such that  $0 \leq h \leq 1$  and in addition

$$h(x) = \begin{cases} 0, & \text{for } |x| \geq 4, \text{ and } |x| \leq 1, \\ 1, & \text{for } 3 \geq |x| \geq 2, \end{cases}$$

Sketch the dynamics of the  $u(0, t)$  as  $t$  changes from zero to infinity.

b) Let  $x \in R^2$ , let  $v(x, t)$  be a solution of the IVP

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \mathbb{R}^2 \times (0, \infty) \\ v(x, 0) = 0 \\ v_t(x, 0) = h(x) \end{cases}$$

Assume  $h(x)$  is a smooth function such that  $0 \leq h \leq 1$  and in addition

$$h(x) = \begin{cases} 0, & \text{for } |x| \geq 4, \text{ and } |x| \leq 1, \\ 1, & \text{for } 3 \geq |x| \geq 2, \end{cases}$$

Sketch the dynamics of the  $v(0, t)$  as  $t$  changes from zero to infinity.

c) Explain if there are major differences between that two dynamics in parts a) and b) and justify your explanation.

Problem 5.

a. Compute weak derivative of the function

$$f(x) = \begin{cases} \frac{2}{\pi}x, & 0 \leq x \leq \frac{\pi}{2} \\ \sin x, & \frac{\pi}{2} \leq x \leq \pi. \end{cases}$$

b. Show that the function  $f(x, y) = \ln \ln \frac{1}{r}$  belong  $W^{1,2}(B(0; R))$ . Here  $r = x^2 + y^2$ ,  $B(0, R)$  is a disk, and  $R < 1$ .