FALL 2008 ODE/PDE PRELIMINARY EXAM

Do 3 problems from Part I and 3 problems from Part II. You must clearly indicate which 6 problems are to be graded.

PART I: ODE

1. Suppose f is continuous and non-negative on $a \le x \le b$ and satisifes

$$f(x) \le \int_0^x \phi(\xi)\, f(\xi)\, d\xi$$

for some positive and continuous ϕ. Then prove that $f \equiv 0$ on the interval $[a, b]$.

2. Consider the initial-value problem for the damped non-linear pendulum $\left\{ \begin{array}{l} y'' + \alpha y' + \sin(y) = 0, \quad \alpha > 0, \\ y(0) = y_0, \quad y'(0) = v_0 \end{array} \right.$.

   A. Explain why the Picard-Linelöf Theorem guarantees the existence of a unique C^2-solution $\varphi(t)$ defined in a neighborhood of 0 with $\varphi(0) = y_0'$ and $\varphi'(0) = v_0$.

   B. Prove that the solution exists for all t.

3. Consider the following linear systems

$$x'(t) = A(t)x(t), \qquad \text{(LH)}$$

$$x'(t) = A(t)x(t) + f(t),\ x(t_0) = x_0 \qquad \text{(LNH)}$$

where $x(t), f(t) \in \mathbb{R}^n$, $A(t)$ is a real $n \times n$ matrix, and $A(t)$, $f(t)$ are continuous on an open interval I containing t_0.

   (a) Define what is meant by a fundamental matrix of (LH), explain why it exists, and derive a formula for the solution of (LNH) subject to the initial condition $x(t_0) = x_0$.

   (b) Prove that the unique solution of (LNH) exists on the whole interval I, whether it be finite or infinite.

4. (A) For an autonomous system $x' = f(x)$, $f \in C^1(\mathbb{R}^n)$, carefully state the definition of a Lyapunov function and state the main Lyapunov Stability Theorem.

   (B) Construct a Lyapunov function to show that the origin is an asymptotically stable equilibrium for $\left\{ \begin{array}{l} x' = -y - x^3 \\ y' = x - y^3 \end{array} \right.$.

5. Consider the system $\left\{ \begin{array}{l} \dot{x} = -y + x(r^4 - 3r^2 + 1) \\ \dot{y} = x + y(r^4 - 3r^2 + 1) \end{array} \right\}$ where $r^2 = x^2 + y^2$ and $\dot{\varphi} = d\varphi/dt$.

   (A) Show that the origin is an unstable focus for this system by constructing an appropriate Lyapunov function. Use the Poincaré-Bendixson Theorem to show there is a periodic orbit in the annular region $A_2 = \left\{ \boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,:\, 0 < \|\boldsymbol{x}\| < 1 \right\}$.

   (B) Show there is a periodic orbit in the annular region $A_2 = \{\boldsymbol{x} \in \mathbb{R}^2 \,:\, 1 < \|\boldsymbol{x}\| < 2\}$.

   (C) Find the unstable and stable limit cycles of this system.

PART II: PDE

1. (A) Let $\Delta u = -F$ in Ω, where Ω is an open bounded region in $\mathbb{R}^2$ with a regular boundary. Suppose $F \in C(\overline{\Omega})$ and is nonpositive in Ω. If $u \in C(\overline{\Omega})$, then $\max_{\boldsymbol{x}\in\Omega} u(\boldsymbol{x}) \le \max_{\boldsymbol{x}\in\partial\Omega} u(\boldsymbol{x})$.

   (B) For the inhomogeneous Dirichlet problem

$$\left\{ \begin{array}{l} \Delta u = -F \quad \text{in} \quad \Omega \subset B(0,R) \\ u = f \quad \text{on} \quad \partial\Omega, \;\; f \in C(\partial\Omega) \end{array} \right\}$$

   (where $B(0,R)$ be the open ball in $\mathbb{R}^2$, centered at $\boldsymbol{x} = 0$, with radius R) show that

$$|u(\boldsymbol{x})| \le \max_{\boldsymbol{x}\in\partial\Omega} |f(\boldsymbol{x})| + \frac{1}{4} \max_{\boldsymbol{x}\in\Omega} |F(\boldsymbol{x})|.$$

2. Solve $uu_x + u_y = 1$ in $\mathbb{R}^2$ for $u = u(x,y)$ with $u(x,x) = \frac{1}{2}x$.

3. Recall that the solution to the initial value heat problem

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \quad x \in \mathbb{R}, \quad t > 0 \\ &u(x,0) = f(x) \end{aligned} \qquad (*)$$

   is given by $u(x,t) = \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{-\infty}^{\infty} e^{-|x-y|^2/(4t)}\, f(y)\, dy.$

   (A) Prove that the solution depends continuously on the data in the sense that if $|f(x) - \widetilde{f}(x)| < \epsilon$, for $-\infty < x < \infty$ then the corresponding solutions satisfy

$$|u(x,t) - \widetilde{u}(x,t)| < \epsilon, \text{ for } -\infty < x < \infty$$

   where u and $\widetilde{u}$ are solutions to $(*)$ corresponding to initial data f and $\widetilde{f}$ respectively.

   (B) Assume $f(x)$ is continuous and bounded. Show that $\lim_{t\to 0^+} u(x,t) = f(x)$.

4. Let $f, f', g \in L^2(0,\pi)$ and consider the following initial-boundary value problem

$$\begin{array}{l} u_{tt}(x,t) = u_{xx}(x,t), \quad 0 < x < \pi, \quad t > 0 \\ u(0,t) = u(\pi,t) = 0, \quad t > 0 \\ u(x,0 = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < \pi \end{array}.$$

   (A) Let $E(t) = \dfrac{1}{2}\displaystyle\int_0^{\pi} \left(u_t^2 + u_x^2\right) dx$ denote the energy. Show that the energy is conserved.

   (B) Calculate $E(t)$, $t > 0$, when $g(x) = 0$ and $f(x) = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \pi/2 \le x \le \pi \end{cases}$.

5. Let $B(0,a)$ be the open ball in $\mathbb{R}^n$, centered at $\boldsymbol{x} = 0$, with radius a. Further, let $u \in C^2(B(0,a)$, $u \in C^0(\overline{B(0,a)})$, $u \ge 0$, $\Delta u = 0$ in $B(0,a)$. Show that for $\|\xi\| < a$,

$$\frac{a^{n-2}(a - \|\xi\|)}{(a + \|\xi\|)^{n-1}} \le u(\xi) \le \frac{a^{n-2}(a + \|\xi\|)}{(a - \|\xi\|)^{n-1}}.$$