SPRING 2009 ODE/PDE PRELIMINARY EXAM

DO 3 PROBLEMS FROM PART I AND 3 PROBLEMS FROM PART II. YOU MUST CLEARLY INDICATE WHICH 6 PROBLEMS ARE TO BE GRADED.

PART I: ODE

1. Consider the general first order homogeneous linear system:

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t), \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $A(\cdot)$ is a continuous $n \times n$ matrix valued function, and $t \in \mathbb{R}$.

(a) Define the *fundamental matrix* for (1). Write the differential equation satisfied by the fundamental matrix. State (without proof) the variation of parameters formula for the solution of the initial value problem

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

(b) Compute an explicit expression for the solution of

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \frac{t}{1+t^2} & 1 \\ 0 & \frac{-4t}{1+t^2} \end{bmatrix} \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Show that the solution goes to zero as $t \to \infty$, regardless of $\mathbf{x}_0$.

2. Consider the general first order homogeneous linear system:

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $A(\cdot)$ is a continuous $n \times n$ matrix valued function, and $t \in \mathbb{R}$.

(a) If $A(t)$ is T-periodic, then state precisely (without proof) the *Floquet decomposition* of the fundamental matrix. What is a necessary and sufficient condition on the fundamental matrix for the existence of an initial state $\mathbf{x}_0$ such that the solution of (2) is T-periodic?

(b) Show that $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$ has an unbounded solution, where

$$A(t) = \begin{bmatrix} \frac{1}{2} - \cos(t) & 12 \\ 147 & \frac{3}{2} + \sin(t) \end{bmatrix}. \tag{3}$$

3. Let $f(t, \mathbf{x})$ be piecewise continuous in t, and locally Lipschitz in $\mathbf{x}$ on $[t_0, t_1] \times D$, for some domain $D \subset \mathbb{R}^n$. Let W be a compact subset of D. Let $\mathbf{x}(t)$ be the solution of $\dot{\mathbf{x}}(t) = f(t, \mathbf{x})$ starting at $\mathbf{x}(t_0) = \mathbf{x}_0 \in W$. Suppose that $\mathbf{x}(t) \in W$ for all $t \in [t_0, T)$, where $T < t_1$.

(a) Show that $\mathbf{x}(t)$ is uniformly continuous on $[t_0, T)$.

(b) Show that $\mathbf{x}(T)$ is defined and belongs to W, and $\mathbf{x}(t)$ is a solution on $[t_0, T]$.

(c) Show that there exists a $\delta > 0$ such that the solution is defined on $[t_0, T + \delta]$.

4. (a) Show that the origin is globally asymptotically stable for the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1^3 - x_2^3. \tag{4}$$

State clearly any theorem you use to prove your claim.

(b) Does the system

$$\ddot{y} + y = \epsilon \, \dot{y} \, (1 - y^2 - \dot{y}^2), \tag{5}$$

have a periodic orbit for $\epsilon > 0$? State clearly any theorem you use to prove your claim. *Hint:* Use the Poincare-Bendixson criterion.

PART II: PDE

1. Let $U \subset B(0, 1)$, where $B(0, 1) = \{(x, y) : x^2 + y^2 < 1\}$, and $(0, 0) \in U$. Let $u(x, y)$ be a solution to the problem

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = -A, \text{ in } U; \quad u = 0 \text{ on } \partial U,$$

where $A > 0$ is a constant. Find an upper bound for $u(0, 0)$, depending on the constant A.

2. See Figure 1. Use a energy method to prove there is at most one smooth solution of the initial/boundary value problem for the heat equation with mixed boundary conditions,

$$
\begin{aligned}
& u_t - \Delta u = f \ \text{ in } \Omega_T = \Omega \times (0, T]; \\
& u = h(x) \ \text{ on } \Gamma_1 \times (0, T], \\
& \frac{\partial u}{\partial \nu} = \nabla u \cdot \nu = f(x) \ \text{ on } \Gamma_2 \times (0, T], \\
& \Gamma_1 \cup \Gamma_2 = \partial \Omega, \quad \Gamma_1 \cap \Gamma_2 = \emptyset \\
& u = g \ \text{ on } \Omega \times \{t = 0\},
\end{aligned}
$$

where ν is the unit outer normal to Ω, Γ_1 and Γ_2 are compact smooth surfaces (See Figure 1).

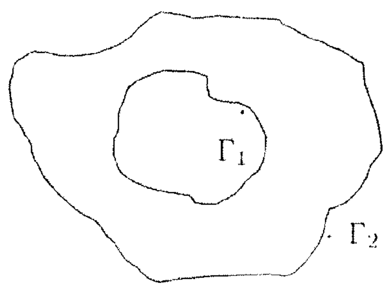

Figure 1: Figure for PDE problem 2.

3. Let $u(x)$ be a solution of the Cauchy problem for the wave equation,

$$
\begin{aligned}
& u_t - \Delta u = 0 \ \text{ in } \mathbb{R}^3 \times (0, \infty); \\
& u = 0, \ u_t = h(x) \ \text{ on } \mathbb{R}^3 \times \{t = 0\}.
\end{aligned}
$$

Suppose the function h is smooth, and $h(x) = 0$ for all x such that $\|x\| > R_0 > 0$. Prove:

(a) There exists t_0 such that $u(0, t) = 0$ for all $t > t_0$,

(b) Find the relation between such a t_0 described in (a) and R_0.

4. Let $U \subset B(0, R_0) \subset \mathbb{R}^n$, where $B(0, R_0) = \{x \mid \|x\| \leq R_0\}$, and suppose $0 \in U$. Let $f(x) = \frac{u(x)}{\|x\|^b}$.
Find conditions on the function $u(x)$, the constant $b > 0$, and n, which guarantee that the function f belongs to the Sobolev space: $W_2^1(U)$.