FALL 2010 ODE/PDE PRELIMINARY EXAM

DO 3 PROBLEMS FROM PART I AND 3 PROBLEMS FROM PART II. YOU MUST CLEARLY INDICATE WHICH 6 PROBLEMS ARE TO BE GRADED. GIVE A FULL STATEMENT OF THE THEOREMS USED IN PROVING THE RESULTS.

PART I: ODE

1. State the Contraction Mapping Theorem. Using this theorem, prove the following result.

   Consider the initial value problem (IVP)

   $$\frac{dx(t)}{dt} = f(t, x(t)); \quad x(t_0) = x_0,$$

   where $t \in [t_0, t_1] \subset \mathbb{R}$ and $x, x_0 \in \mathbb{R}^n$. Let $f(t, x)$ be piecewise continuous in t and satisfy the Lipschitz condition

   $$\|f(t,x) - f(t,y)\| \leq L\|x - y\|$$

   for all $x, y \in B = \{x \in \mathbb{R}^n \,|\, \|x - x_0\| \leq r\}$ for some $r, L > 0$ and for all $t \in [t_0, t_1]$. Then, there exists some $\delta > 0$ such that IVP has a unique solution over $[t_0, t_0 + \delta]$.

2. Prove the result: There exists an $x_0 \in \mathbb{R}^n$ such that the solution of:

   $$\frac{dx(t)}{dt} = A(t)\,x(t) + f(t), \quad x(t_0) = x_0$$

   is τ-periodic if and only if $f(t)$ is such that

   $$\int_{t_0}^{t_0+\tau} z^T(t) f(t)\, dt = 0$$

   for all τ-periodic solutions $z(t)$ of the adjoint equation

   $$\frac{dz(t)}{dt} = -A^T(t)\,z(t), \quad z(t_0) = z_0.$$

3. Consider the two dimensional system

$$\begin{aligned}\dot{x} &= \lambda x - y - xr^2 + \lambda \frac{x^3}{r^3} \\ \dot{y} &= x + \lambda y - yr^2 + \lambda \frac{x^2 y}{r^3},\end{aligned}$$

where $r = +\sqrt{x^2 + y^2}$. Prove that this system has a stable limit cycle when $\lambda > 0$.

4. Consider the three dimensional system:

$$\begin{aligned}\dot{x} &= \frac{1}{1 + z^m} - a\,x \\ \dot{y} &= x - b\,y \\ \dot{z} &= y - c\,z\end{aligned}$$

where $a, b, c > 0$ and m is a positive integer.

(a) Show that there exists an equilibrium point (x_0, y_0, z_0) in the first octant (that is, $x_0, y_0, z_0 > 0$).

(b) Show that there exist positive numbers X, Y, Z large enough such that the cube $D = \{(x, y, z) \,|\, 0 \le x \le X,\ 0 \le y \le Y,\ 0 \le z \le Z\}$ is positively invariant for the system.

(c) For $m = 1$, show that the equilibrium point is asymptotically stable. You may use the following result: The necessary and sufficient condition for the cubic polynomial $p(s) = \alpha_0 s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$ to have roots with negative real parts is (i) $\alpha_i > 0$ for $i = 0, 1, 2, 3$, and (ii) $\alpha_1 \alpha_2 > \alpha_0 \alpha_3$.

5. (15 points) Consider the two sample problem where the responses are lifetimes. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent samples from $F_X(x)$ and $F_Y(y)$, respectively. The hazard function for X is defined by

$$h_X(t) = \frac{f(t)}{1 - F(t)}$$

and similarly for Y. Suppose that $1 - F_Y(y) = ((1 - F_X(y))^\alpha$, $\alpha > 0$.

(a) Show that $h_Y(t) = \alpha h_X(t)$.

(b) Show that $P_\alpha(Y > X) = (\alpha + 1)^{-1}$.

(c) Show that $E_\alpha(S) = \frac{mn}{1+\alpha}$, where S is the Mann-Whitney-Wilcoxon statistic defined as

$$S = \sum_{i=1}^{n} \sum_{j=1}^{m} I(Y_i > X_j).$$

(d) Use (b) and (c) to find a method of moments estimator for α.

6. (15 points) Suppose $X_1, \ldots, X_n$ are independent discrete random variables with probability mass function for X_i, $i = 1, \ldots, n$, given by

x	1	2	3
$p(x;\theta)$	θ^2	$2\theta(1-\theta)$	$(1-\theta)^2$

with $0 < \theta < 1$. Let N_i be the number of observations equal to i, $i = 1, 2, 3$.

(a) Find the form of the most powerful test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$. (Hint: the MP test depends on $U = 2N_1 + N_2$.)

(b) Show that the test as in (a) is also UMP for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$.

(c) When $n = 2$, find the UMP test of size $\frac{1}{16}$ for $H_0 : \theta = 1/2$ versus $H_1 : \theta > 1/2$.

7. (15 points) Suppose that $X_1, \ldots, X_n$ are IID $N(0, \sigma^2)$ and that $Y_1, \ldots, Y_m$ are IID $N(0, \tau^2)$. Assume further that X_i and Y_j are independent for any i and j.

(a) Find the MLE of σ^2 and τ^2.

(b) Construct the LRT statistic for testing $H_0 : \sigma^2 = \tau^2$ against $H_1 : \sigma^2 \neq \tau^2$.

(c) Show that the LRT statistic can be written in such a way that it involves the data only through the statistic

$$G = \frac{n \sum_{i=1}^{m} Y_i^2}{m \sum_{j=1}^{n} X_j^2}.$$

(d) Find the distribution of G under H_0.

(e) Construct the size 0.05 LRT test.

PART II: PDE

1. Let $U \subset \mathbb{R}^n, n > 2$ be a bounded, simply connected domain with a smooth boundary. Assume $0 \in U$. Let $u \in C^2(U \setminus 0) \cap C^0(\overline{U} \setminus 0)$ be a solution of the problem:

$$\Delta u = 0 \text{ in the domain } U \setminus 0 \tag{1}$$

$$u = 0 \text{ on the boundary } \partial U \tag{2}$$

Let

$$M(r) = \sup_{|x|=r} |u|.$$

(A) Prove that if

$$\lim_{r \to 0} M(r) r^{n-2} = 0, \tag{3}$$

then $u(x) = 0$ for all $x \in U$.

(B) Construct a bounded, simply connected domain U and a particular solution u to equations (1) and (2) such that $u \neq 0$ in the interior of U.

2. Let $U \subset \mathbb{R}^n$ be a bounded, simply connected domain with a smooth boundary. Consider the initial boundary value problem

$$\begin{aligned} \Delta u &= u_t - h(x,t) \text{ in } D_T = U \times (0,T], \\ u &= f_1(x,t) \text{ on } \Gamma_1 \times (0,T], \\ \frac{\partial u}{\partial \nu} &= f_2(x,t) \text{ on } \Gamma_2 \times (0,T], \\ u(x,0) &= a(x). \end{aligned}$$

Here $\Gamma_1 \cup \Gamma_2 = \partial U$ is a smooth surface, and $\Gamma_1 \cap \Gamma_2 = \emptyset$ and ν is the unit outward vector on the boundary. Let C_1^2 denote the set of functions having two continuous derivatives in x and one continuous derivative in t.

Using the energy method prove that there exists at most one solution $u(x,t) \in C_1^2(\overline{D}_T)$ for the initial boundary value problem.

3. Let $u(x,t) \in C^2(\mathbb{R} \times [0,\infty))$ solves the initial value problem

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(x,0) &= g(x) \\ u_t(x,0) &= h(x). \end{aligned}$$

Suppose $g(x)$ and $h(x)$ have compact support.

Prove that

$$\int_{-\infty}^{\infty} (u_x^2(x,t) + u_t^2(x,t))dx$$

is a constant.

4. Let $U \subset \mathbb{R}^n$ be a bounded, simply connected domain with a smooth boundary. Let $u(x) \in C^1(\overline{U})$ and $u(x) = 0$ on ∂U.

Prove the Poincaré inequality:

$$\int_U |\nabla u(x)|^2 dx \geq C_0 \int_U u^2(x) dx,$$

where $C_0 > 0$ is a constant depending only on the domain U.