SPRING 2010 ODE/PDE PRELIMINARY EXAM

DO 3 PROBLEMS FROM PART I AND 3 PROBLEMS FROM PART II. YOU MUST CLEARLY INDICATE WHICH 6 PROBLEMS ARE TO BE GRADED. GIVE A FULL STATEMENT OF THE THEOREMS USED IN PROVING THE RESULTS.

PART I: ODE

1. Consider the general first order homogeneous linear system:

$$\dot{X}(t) = AX(t) - X(t)A, \qquad X(0) = X_0 \tag{1}$$

where $X(t) \in \mathbb{R}^{n\times n}$ for each $t \in \mathbb{R}$ and A is a fixed $n \times n$ real matrix.

(a) Find an expression for the solution $X(t)$ in terms of the matrices A and X_0 for an arbitrary $t \in \mathbb{R}$.

(b) Show that the eigenvalues of $X(t)$ for any t are the same as the eigenvalues of X_0.

(c) When A is skew-symmetric (that is, $A^T = -A$), show that the system (1) is stable. *Hint*: Use the fact that the 2-norm of a $n \times n$ real matrix B is equal to the spectral radius of B^TB, that is, $\|B\|_2 = [\max_{1\leq i\leq n} |\lambda_i(B^TB)|]^{\frac{1}{2}}$.

2. Consider the general first order homogeneous linear system:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $t \in \mathbb{R}$, and A is given by: $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, where A_{11} is a $r \times r$ matrix with $1 \leq r < n$.

(a) Show that the fundamental matrix $X(t)$ is of the form

$$X(t) = \begin{bmatrix} X_{11}(t) & X_{12}(t) \\ 0 & X_{22}(t) \end{bmatrix}, \tag{3}$$

where X_{11} is a $r \times r$ matrix. Find expressions for $X_{ij}(t)$ in terms of A_{ij}.

2. (b) Show that if $\mathbf{x}_0 \in \text{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_r\}$, then $\mathbf{x}(t) \in \text{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_r\}$ for all t.

(c) Suppose A has real eigenvalues. Then the Schur decompostion of A is given by $A = PTP^T$ where P is orthogonal and T is an upper triangular matrix. Let $P = [\mathbf{p}_1 \ \mathbf{p}_2 \ \cdots \ \mathbf{p}_n]$. Show that if $\mathbf{x}_0 \in \text{span}\{\mathbf{p}_1, \cdots, \mathbf{p}_r\}$, then $\mathbf{x}(t) \in \text{span}\{\mathbf{p}_1, \cdots, \mathbf{p}_r\}$ for all t, for any $1 \leq r \leq n$.

3. Let $f(t, \mathbf{x})$ be piecewise continuous in t, and locally Lipschitz in $\mathbf{x}$ on $[t_0, t_1] \times D$, for some domain $D \subset \mathbb{R}^n$. Let W be a compact subset of D. Let $\mathbf{x}(t)$ be the solution of $\dot{\mathbf{x}}(t) = f(t, \mathbf{x})$ starting at $\mathbf{x}(t_0) = \mathbf{x}_0 \in W$. Suppose that $\mathbf{x}(t) \in W$ for all $t \in [t_0, T)$, where $T < t_1$.

(a) Show that $\mathbf{x}(t)$ is uniformly continuous on $[t_0, T)$.

(b) Show that $\mathbf{x}(T)$ is defined and belongs to W, and $\mathbf{x}(t)$ is a solution on $[t_0, T]$.

(c) Show that there exists a $\delta > 0$ such that the solution is defined on $[t_0, T + \delta]$.

4. Consider the system:

$$\begin{aligned}
\frac{dx_1}{dt} &= f(x_3) - b_1 x_1 \\
\frac{dx_2}{dt} &= a_1 x_1 - b_2 x_2 \\
\frac{dx_3}{dt} &= a_2 x_2 - b_3 x_3,
\end{aligned}$$

where a_1, a_2, b_1, b_2, b_3 are positive real numbers, and f is a positive and monotone decreasing function. Assume f to be continuously differentiable.

(a) Show that there exists an equilibrium point $\mathbf{x}_0 = (x_{01}, x_{02}, x_{03})$ in the first octant (that is, $x_{0i} > 0$ for $i = 1, 2, 3$).

(b) Show that there exists $\epsilon > 0$ such that if $-\epsilon < f'(x_{03}) < 0$ then the equilibrium point is asymptotically stable.

PART II: PDE

1. Let $$Lu = \sum_{i=1}^{n} a_i(x)\, \frac{\partial^2 u}{\partial x_i^2}.$$

   Assume that the coefficients of the operator L satisfy the conditions $0 < \alpha \le a_i(x) \le \beta < \infty$ for $i = 1, 2, \cdots, n$, where α, β are constants. Let $U \subset \mathbb{R}^n$ be the domain $(n > 2)$.

   (a) Prove that if $\frac{\alpha}{\beta} > \frac{2}{n}$, then there exist $s > 0$ such that the function $G(x) = |x|^{-s}$ is super-elliptic, that is $LG(x) \le 0$.

   (b) Let $u \in C^2(U) \cap C^0(\overline{U})$ be sub-elliptic (that is, $Lu \ge 0$), and $\nu \in C^2(U) \cap C^0(\overline{U})$ be super-elliptic (that is, $L\nu \le 0$). State (you need not prove) the maximum principle for super-elliptic and sub-elliptic functions.

   (c) Let $Lu = 0$ in $U \setminus \{0\}$, $u \in C^2(U \setminus \{0\}) \cap C^0(\overline{U} \setminus \{0\})$. Here $0 \in U$. Let $M(r) = \max_{|x|=r} |u(x)|$ for $r > 0$.

   Assume $\frac{\alpha}{\beta} > \frac{2}{n}$, and $\lim_{r \to 0} M(r) r^s = 0$ and $u(x) = 0$ on the boundary ∂U, where s is from item (a) and is such that $LG \le 0$.

   Prove that $u(x) = 0$ in U.

2. Suppose that the domain $U \subset \mathbb{R}^n$ has a smooth boundary, and $D = U \times (0, T]$. Let $a_i(t) > 0$ for $i = 1, 2, \cdots$, $\beta(x,t) > \beta_0 > 0$ for all $x \in U$ and $t > 0$. Let $u(x,t) \in C^2(D) \cap C^0(\overline{D})$ be a solution of the intial-boundary value problem (IBVP):

$$\beta(x,t)\, \frac{\partial u}{\partial t} = \sum_{i=1}^{n} a_i(t)\, \frac{\partial^2 u}{\partial x^2} - u \qquad \text{in } U \times (0, T],$$
$$u = h(x,t) \qquad \text{on } \partial U \times (0, T],$$
$$u = g(x) \qquad \text{on } \partial U \times \{0\},$$

   where $h(x,t)$ and $g(x)$ are given data.

   Prove that the solution $u(x,t)$ of the above IBVP is unique.

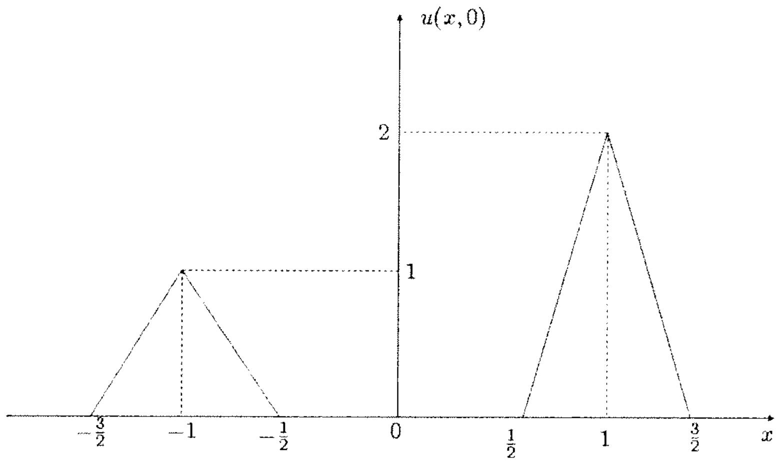

Figure 1: Figure for PDE problem 3 (figure not to scale).

3. Let $u(x,t)$ be a displacement of the infinitely long string (assume $-\infty < x < \infty$). Assume that at time $t = 0$, the displacement of the string is perturbed as shown in Figure 1. Assume that the velocity at $t = 0$ is equal to zero.

   (a) State the above problem in terms of an initial value problem for the wave equation
   $$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}.$$

   (b) Find a time t_0 and a point x_0 with $-\frac{1}{2} < x_0 < \frac{1}{2}$, so that the displacement attains its maximum value.

4. (a) Let $f(\mathbf{x})$, $\mathbf{x} = (x_1, \cdots, x_n)$ be a function belonging to $L^2(U)$. State the definition of generalized derivative f_{x_i}; $1 \le i \le n$ in Sobolev sense.

   (b) Define the Sobolev space $W_2^1(U)$.

   (c) Prove the Poincare' inequality for the function $u(x_1, x_2) \in C^1(\overline{U})$ satisfying the following condition: $u(x_1, x_2) = 0$ when $x_2 = 1 - x_1$. Here, $U = \{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < 1 - x_1\}$ is a triangle.