FALL 2011 ODE/PDE PRELIMINARY EXAM

Do 3 problems from Part I and 3 problems from Part II. You must clearly indicate which 6 problems are to be graded. Give a full statement of the theorems used in proving the results. Write on one side of the paper only.

PART I: ODE

1. A function $g : D \to \mathbb{R}^n$ where D is an open set in $\mathbb{R}^n$ is said to be F-differentiable at $x \in D$ if there is a $n \times n$ matrix A such that

$$\lim_{h \to 0} \frac{1}{\|h\|} \|g(x+h) - g(x) - Ah\| = 0. \tag{1}$$

The matrix A is denoted as $g'(x)$ and is called the F-derivative of g at x.

(a) Suppose $g : D \to \mathbb{R}^n$ is F-differentiable at $x = 0 \in D$ and satisfies:

$$\lim_{x \to 0} \frac{\|g(x)\|}{\|x\|} = 0. \tag{2}$$

Prove that $g(0) = 0$ and $g'(0) = 0$. *Hint:* Rewrite (1) and (2) in $\epsilon - \delta$ form.

(b) Suppose that $g : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuously F-differentiable on D, where $0 \in D$, and g satisfies (2). Continuous F-differentiability on D means that $g'(x)$ is a matrix with entries that are continuous functions of $x \in D$. Prove that there exists $r > 0$ and $\delta > 0$ such that $\forall x, x' \in B_r(0)$, $\|g(x) - g(x')\| \leq \delta \|x - x'\|$. *Hint:* Use part(a) and the continuous F-differentiability assumption.

(c) State a basic theorem yielding existence and uniqueness of solutions for the system: $\dot{x}(t) = f(x(t)), \quad x(0) = x_0$, including any assumptions on $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$.

(d) Suppose that $g : D \to \mathbb{R}^n$ is continuously F-differentiable on D, where $0 \in D$, and satisfies (2). Prove that there exists $\epsilon > 0$ such that the system:

$$\dot{x} = Ax + g(x), \quad x(0) = x_0 \tag{3}$$

has a unique solution for any $x_0 \in D$ and on the interval $(-\epsilon, \epsilon)$.

2. (a) State La Salle's stability theorem.

(b) If $k > 0$ is a constant, is $x = 0$ stable or unstable for $\dot{x} = xk^2 - x^3$?

(c) Is $(x, k) = (0, 0)$ a stable equilibrium for the system $\dot{x} = xk^2 - x^3$, $\dot{k} = 0$.

3. (a) State the Comparison Lemma.

(b) Consider the system:

$$\begin{aligned}\dot{x}_1 &= -\frac{1}{\tau}x_1 + \tanh(\lambda x_1) - \tanh(\lambda x_2)\\ \dot{x}_2 &= -\frac{1}{\tau}x_2 + \tanh(\lambda x_1) + \tanh(\lambda x_2),\end{aligned}$$

where λ and τ are positive constants. Using the fact that $-1 < \tanh(u) < 1$ for all real u, show that the function $r = \sqrt{x_1^2 + x_2^2}$ satisfies the differential inequality

$$\dot{r} \leq -\frac{1}{\tau}r + 2\sqrt{2}.$$

(c) Using the comparison lemma or otherwise, show that the solution $x(t) = (x_1(t), x_2(t))$ satisfies the inequality

$$\|x(t)\|_2 \leq e^{-\frac{t}{\tau}}\|x(0)\|_2 + 2\sqrt{2}\tau(1 - e^{-\frac{t}{\tau}}).$$

4. Consider the general first order homogeneous linear system:

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0, \tag{4}$$

where $x(t) \in \mathbb{R}^n$, $A(\cdot)$ is a continuous, $n \times n$ matrix valued function, and $t \in \mathbb{R}$.

(a) If $A(t)$ is T-periodic, then state precisely (without proof) the *Floquet decomposition* of the fundamental matrix. What is the necessary and sufficient condition on the fundamental matrix for the existence of an initial state x_0 such that the solution of (4) is T-periodic?

(b) Show that $\dot{x}(t) = A(t)x$ has an unbounded solution, where

$$A(t) = \begin{bmatrix} \frac{1}{2} - \cos(t) & 12 \\ 147 & \frac{3}{2} + \sin(t) \end{bmatrix}. \tag{5}$$

PART II: PDE

1. State and prove the maximum principle for solutions of Laplace's equation, that is, the equation $\Delta u = 0$.

2. Let Ω be an open, bounded subset of $\mathbb{R}^n$ with C^1 boundary. Let the function $u(x,t)$ be a classical solution of the initial boundary value problem

$$u_t(x,t) - \Delta u(x,t) + b|\nabla u(x,t)| = 0, \quad x \in \Omega, \quad t > 0,$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0,$$

$$u(x,0) = u_0(x), \quad x \in \Omega,$$

where b is a constant, and $u_0 \in C(\overline{\Omega})$ is a given initial data.

Prove that there is at most one such solution $u(x,t)$.

3. Let $u(x,t)$ be defined on $\mathbb{R} \times [0,\infty)$ and solve the problem

$$u_{tt}(x,t) - u_{xx}(x,t) = f(x,t), \quad x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = 0,\ u_t(x,0) = 0, \quad x \in \mathbb{R},$$

where $f \in C^1(\mathbb{R} \times [0,\infty))$ is a given function.

Show that for any $x \in \mathbb{R}$, $t > 0$, we have

$$|u(x,t)| \le \frac{t^2}{2} \cdot \sup\{\, |f(x,t)| : (x,t) \in \mathbb{R} \times [0,\infty)\,\}.$$

4. Find the entropy solution $u(x,t)$ of the following initial-value problem

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = \begin{cases} 3, & x < -1 \\ -1, & -1 < x < 0 \\ 2, & x > 0. \end{cases}$$