SPRING 2011 ODE/PDE PRELIMINARY EXAM

DO 3 PROBLEMS FROM PART I AND 3 PROBLEMS FROM PART II. YOU MUST CLEARLY INDICATE WHICH 6 PROBLEMS ARE TO BE GRADED. GIVE A FULL STATEMENT OF THE THEOREMS USED IN PROVING THE RESULTS.

PART I: ODE

1. Consider the linear time invariant system of equations:

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2, \end{aligned} \tag{1}$$

where A_{11} is an $n_1 \times n_1$ matrix and A_{22} is an $n_2 \times n_2$ matrix.

(a) State La Salle's invariance principle.

(b) Suppose there exists an $n_1 \times n_1$ constant matrix P such that

$$A_{11}^T P + P A_{11} = -I. \tag{2}$$

Prove that P must be non-singular.

(c) Suppose that there exists a constant matrix P satisfying (2). Use the function $V(x_1) = x_1^T P x_1$ to prove that the origin $x_1 = 0$ is locally asymptotically stable for the system

$$\dot{x}_1 = A_{11}x_1. \tag{3}$$

(d) Suppose that there exists a P satisfying (2). Further suppose that there exists an $n_2 \times n_2$ constant matrix Q such that

$$A_{22}^T Q + Q A_{22} = -I. \tag{4}$$

Use $V(x_1, x_2) = x_1^T P x_1 + x_2^T Q x_2$ to prove that the origin $(x_1, x_2) = (0, 0)$ for the system (1) is locally asymptotically stable provided $\|A_{12}\|$ and $\|A_{21}\|$ are sufficiently small.

2. Consider the two dimensional system:

$$\begin{aligned}\frac{dx}{dt} &= -y + x(1 - x^2 - y^2) \\ \frac{dy}{dt} &= x + y(1 - x^2 - y^2).\end{aligned}$$

(a) Show that the set $S = \{(x, y) \mid x^2 + y^2 = 1\}$ is an invariant set for the system. This means that if the initial point (x_0, y_0) is in the set S, then the solution trajectory lies in S.

(b) Show that for an initial condition in S the solution is of the form: $x(t) = \cos(\omega t + \phi)$ and $y(t) = \sin(\omega t + \phi)$, where $\omega > 0$ and $\phi \in [0, 2\pi)$. What is the value of ω?

(c) Obtain the equations of the system in polar coordinates. Using these equations, show that the set S is an ω limit set - that is, every trajectory except the zero trajectory spirals towards this set as $t \to \infty$.

3. Consider the ODE:

$$\frac{d^2y}{dt^2}(t) + t^{-1}\, a_1\, \frac{dy}{dt}(t) + t^{-2}\, a_0\, y(t) = 0, \tag{5}$$

where a_0 and a_1 are constants.

(a) Write the ODE in the form:

$$\frac{dx}{dt} = t^{-1}\, A\, x, \tag{6}$$

where A is a constant matrix, and x is a vector.

(b) For a scalar $t > 0$ and a constant square matrix A, define $t^A = e^{A\ln(t)}$. Show that the solution of (6) is of the form: $x(t) = t^A\, x_0$ where $x_0 = x(1)$. *Hint*: Re-parameterize the variable t and rewrite (6) in terms of the new variable.

(c) Prove that for scalars $t_1, t_2 > 0$, we have:

$$t_1^A\, t_2^A = (t_1\, t_2)^A.$$

Hint: Use any properties of the exponential of a matrix you know.

(d) Prove that for $t > 0$:

$$\det(t^A) = t^{\mathrm{trace}(A)}.$$

4. Show that the origin is locally asymptotically stable for the system

$$\dot{x}_1 = -\sin(x_1) + x_1 x_2^2, \quad \dot{x}_2 = -x_2 - x_1^2 x_2. \tag{7}$$

State clearly any theorem you use to prove your claim.

PART II: PDE

5. Given $L > 0$. Let $u_T(x,t)$, $T > 0$, be a classical solution of the initial value problem:

$$(u_T)_t = (u_T)_{xx}, \quad 0 < x < L,\ t > 0,$$

$$u_T(0,t) = u_T(L,t) = 0, \quad t > 0$$

$$u_T(x,-T) = 1, \quad 0 < x < L.$$

(a) Prove that for $t > -T$

$$\frac{d}{dt}\int_0^L u_T^2(x,t)dx \le -\int_0^L (u_T)_x^2(x,t)dx.$$

(b) Prove that there exists a positive constant C such that

$$C\int_0^L u_T^2(x,t)dx \le \int_0^L (u_T)_x^2(x,t)dx.$$

(c) Prove that

$$\int_0^L u_T^2(x,t)dx \to 0 \text{ as } t \to \infty.$$

6. Let U be an open subset of $\mathbb{R}^n$ and let $u \in C^2(U) \cap C(\overline{U})$ be a classical non-negative solution of the equation

$$\Delta u + cu = 0 \quad \text{in } U,$$

where c is a positive number.

(a) Find a function $f(r)$, $r \ge 0$, such that $\lim_{r\to 0} f(r) = 0$ and the function $v(x) = u(x) + Mf(|x|)$ is subharmonic in U, that is, $-\Delta v \le 0$ in U. Here $M = \max_{\overline{U}} u$.

(b) Show that there is $R > 0$ such that if $U \subset \{x \in \mathbb{R}^n : |x| < R\}$ then

$$\max_{\overline{U}} u \le 2 \max_{\partial U} u.$$

7. Let U be an open, bounded subset of $\mathbb{R}^n$ with C^2 boundary. Let $u(x,t)$ be a classical solution of the initial boundary value problem

$$u_{tt} + k(x,t)u_t = \operatorname{div}(a(x)\nabla u), \ x \in U, \ t > 0,$$

$$u = \phi(x,t), \ x \in \partial U, \ t > 0,$$

$$u(x,0) = f(x), \ u_t(x,0) = g(x), \ x \in U,$$

where $k(x,t)$, $a(x)$, $g(x)$, $f(x)$, $\phi(x,t)$ are smooth functions.
Prove that if $k(x,t) \geq 0$ and $0 < c \leq a(x,t) \leq C < \infty$ for all (x,t), then the above solution $u(x,t)$ is unique.

8. Solve the equation

$$u_x u_y = 3 \quad \text{on } U = \{(x,y) \in \mathbb{R}^2 : x > 0\},$$

$$u(x,0) = \log x, \quad x > 0,$$

where $u = u(x,y)$, and log denotes the natural logarithm.