FALL 2012 ODE/PDE PRELIMINARY EXAM

PART I: ODE

Do all four problems.

1. (a) State the Comparison Lemma.

   (b) Use the Comparison Lemma to show that the solution $x(t) = (x_1(t), x_2(t))$ of the system

$$\begin{cases} \dot{x}_1 = -x_1 + \dfrac{2x_2}{1+x_2^2} \\ \dot{x}_2 = -x_2 + \dfrac{2x_1}{1+x_1^2} \end{cases}$$

   satisfies the inequality $||x(t)||_2 \leq e^{-t}||x(0)||_2 + \sqrt{2}(1-e^{-t})$.

2. (a) Show that the following system has no periodic orbits on $\mathbb{R}^2$

$$\begin{cases} \dot{x}_1 = x_1(4 - x_2 - x_1^2) \\ \dot{x}_2 = x_2(x_1 - 1). \end{cases}$$

   (b) Consider the system

$$\begin{cases} \dot{x}_1 = x_2^3 - 4x_1 \\ \dot{x}_2 = x_2^3 - x_2 - 3x_1. \end{cases}$$

   (i) Find all the equilibria and determine their stabilities.

   (ii) Show that any trajectory that starts on the line $x_1 = x_2$ stays on the line for all forward time.

3. Let A be an $n \times n$ constant matrix and consider the linear system $\dot{x} = Ax$.

   (a) Define what is meant by a fundamental matrix and explain why it exists.

   (b) State the definition of stability of the origin.

   (c) Let $X(t)$ be the fundamental matrix for the linear system with $X(0) = I$, the identity matrix. Suppose that there exists $M > 0$ such that $||X(t)|| \leq M$ for $t \geq 0$. Prove that the origin is stable for the linear system.

4. Let $f : D \to \mathbb{R}^n$ be locally Lipschitz, where D is a domain of $\mathbb{R}^n$, $0 \in D$, and $f(0) = 0$.

   (a) State the Liapunov Stability Theorem for the autonomous system $\dot{x} = f(x)$.

   (b) Applying the variable gradient method to find a Liapunov function and show that the origin is asymptotically stable for the autonomous system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -(x_1 + x_2) - \sin(x_1 + x_2). \end{cases}$$

AUGUST 2012. **PRELIMINARY EXAMINATION**

PART II: Partial Differential Equations

Do three out of four problems below. Write in the following boxes the three problems that are to be graded:

1. Let U be an open, bounded subset of $\mathbb{R}^n$ and $b(x)$ be a continuous function from $\bar{U}$ to $\mathbb{R}^n$. Suppose $u \in C^2(U) \cap C(\bar{U})$ is a solution of

$$-\Delta u + b(x) \cdot \nabla u = 0 \text{ in } U.$$

State and prove the maximum principle for $u(x)$.

2. Let U be an open, bounded subset of $\mathbb{R}^n$ with C^1 boundary. Let $T > 0$ and $U_T = U \times (0, T]$. Suppose $u_1, u_2 \in C^2(\bar{U}_T)$ solve the problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T, \\ u = g & \text{on } \partial U \times [0, T], \end{cases}$$

where g is a given function. Define

$$E(t) = \int_U |u_2(x,t) - u_1(x,t)|^2 dx.$$

(a) Prove that the function $f(t) = \ln E(t)$ is convex on an interval I if $E(t) > 0$ on I.

(b) Prove that $u_1 \equiv u_2$ in U_T if $u_1(x, T) = u_2(x, T)$ for all $x \in U$.

3. Let $u = u(x,t) \in C^2([0,\pi] \times [0,T])$ be a solution to the problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left((\sin^2 x)\frac{\partial u}{\partial x}\right) = k\frac{\partial u}{\partial t} \quad \text{in } (0,\pi) \times (0,T),$$

$$u(x,0) = g_0(x), \quad u_t(x,0) = g_1(x) \quad \text{on } (0,\pi),$$

where k is a constant and g_0, g_1 are given functions.

Prove that this solution $u(x,t)$ is unique.

4. Solve the following equation

$$u_x + (u_y)^2 = 2 \quad \text{on} \quad \mathbb{R}^2,$$

$$u(0,y) = \frac{y^2}{2}, \quad y \in \mathbb{R},$$

where $u = u(x,y)$ is a scalar function.