SPRING 2012 ODE/PDE PRELIMINARY EXAM

PART I: ODE

Do all problems.

1. Consider the initial value problem

$$\dot{x} = f(t, x), \;\; x(0) = x_0,$$

where $f(t, x)$ is locally Lipschitz in x, piecewise continuous in t, and satisfies

$$||f(t, x)||_2 \leq a + b||x||_2 \text{ for all } (t, x) \in [0, \infty) \times \mathbb{R}^n$$

for positive constants a and b.

(a) Show that a unique solution exists locally and the solution satisfies

$$||x(t)||_2 \leq ||x_0||_2 e^{bt} + \frac{a}{b}(e^{bt} - 1)$$

for $t \geq 0$ for which the solution is defined locally.

(b) Hence, prove that the solution exists on $[0, \infty)$.

2. Consider the system

$$\begin{cases} \dot{x_1} = -x_2 - x_1^3 \\ \dot{x_2} = x_1^5. \end{cases}$$

(a) Explain why the linear stability theory fails to determine the local stability of the zero steady state solution.

(b) State the LaSalle's invariance principle for a general autonomous system $\dot{x} = f(x)$, where $f(0) = 0$. Clearly state the assumptions in the principle.

(c) Apply a variation of the LaSalle's invariance principle to prove that the origin is asymptotically stable for the given two-dimensional system.

3. There are different methods to show that a two-dimensional autonomous system has a periodic orbit.

(a) State the Poincaré-Bendixson Criterion.

(b) Apply the criterion to show that the system

$$\begin{cases} \dot{x} = \alpha x - y - \alpha x(x^2 + y^2) \\ \dot{y} = x + \alpha y - \alpha y(x^2 + y^2) \end{cases}$$

has a periodic orbit, where $\alpha > 0$ is a parameter.

(c) Prove that the system in (b) has a periodic solution by converting the system to polar coordinates.

4. Let $A = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be a matrix.

(a) Determine the fundamental matrix $X(t)$ for $\dot{x} = Ax$ with $X(0) = I$, the identity matrix.

(b) Derive the formula for variation of parameters.

(c) Use the method of variation of parameters to solve the linear nonhomogeneous system $\dot{x} = Ax + b(t)$, $x(0) = x_0$, where $x_0 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and $b(t) = \begin{pmatrix} \sin(2t) \\ \cos(2t) \\ 1 \end{pmatrix}$.

MAY 2012. **PRELIMINARY EXAMINATION**

PART II: Partial Differential Equations

Do three out of four problems below. Write in the following boxes the three problems that are to be graded:

1. Let $u \in C^2(\mathbb{R}^n)$, where $n \geq 2$, be a solution of the equation

$$-\Delta u(x) + cu(x) = 0, \quad x \in \mathbb{R}^n,$$

where c is a constant. For $r > 0$, let

$$\phi(r) = \fint_{\partial B_r} u dS = \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B_r} u dS,$$

where $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ and α_n is the volume of B_1.

Show that $\phi(r)$ satisfies the ordinary differential equation

$$\phi''(r) + \frac{n-1}{r}\phi'(r) - c\phi(r) = 0, \quad r > 0.$$

2. Consider the equation

$$u_t - \alpha \Delta u + u|\nabla u|^2 = 0,$$

where α is a positive constant, $u = u(x,t)$ with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

State and prove the maximum principle for this equation on a domain $U_T = U \times (0, T]$ where U is an open, bounded subset of $\mathbb{R}^n$ and $T > 0$.

3. Let $u(x,t)$ be defined on $\mathbb{R} \times [0, \infty)$ and solve the problem

$$u_{tt}(x,t) - u_{xx}(x,t) = f(x,t), \quad x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = 0,\ u_t(x,0) = 0, \quad x \in \mathbb{R},$$

where $f \in C^1(\mathbb{R} \times [0, \infty))$ is a given function with compact support.

(a) Show that for any $T > 0$ there exists $L > 0$ such that $u(x,t) = 0$ for all $|x| \geq L$, $0 \leq t \leq T$.

(b) Show that for any $L > 0$ there exists $T > 0$ such that $u(x,t)$ is a constant on $[-L, L] \times [T, \infty)$.

4. Find the entropy solution $u(x,t)$ of the following initial-value problem

$$u_t + 2\,u\,u_x = 0, \quad x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = \begin{cases} 3, & x < 0 \\ -1, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$$