2013 May ODE/PDE Preliminary Examination

Part I: ODE. Do 3 of the following 4 problems. You must clearly indicate which 3 are to be graded. Problem 1, 2, and 3 will be graded if no indication is given. Strive for clear and detailed solutions.

1. a) Let $A(t)$ be a continuous $n \times n$ matrix with the property that

$$A(t)\left(\int_0^t A(s)\ ds\right) = \left(\int_0^t A(s)\ ds\right) A(t).$$

Prove that

$$\Phi(t) = e^{\int_0^t A(s)\ ds}$$

is a fundamental matrix of the system $\dot{x} = A(t)x$.

b) Give a counter example to show that the result is not true if

$$A(t)\left(\int_0^t A(s)\ ds\right) \neq \left(\int_0^t A(s)\ ds\right) A(t).$$

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a global Lipschitz function and $\phi(t, y)$ be the solution of the initial value problem

$$\dot{x} = f(x), \quad x(0) = y.$$

Accept the fact without proof that the maximal interval of existence of $\phi(t, y)$ is $(-\infty, \infty)$ for any y. Prove that for any $a > 0$ and for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $\|y_1 - y_2\| < \delta$ and for all $t \in [-a, a]$

$$\|\phi(t, y_1) - \phi(t, y_2)\| < \epsilon.$$

3. Use a Lyapunov function to show that the origin of the system

$$\begin{aligned} \dot{x}_1 &= x_1^5 + x_2^6 \\ \dot{x}_2 &= -x_2 + x_1^6 \end{aligned}$$

is unstable. State any theorem that you invoke.

4. Prove that the system

$$\begin{aligned} \dot{x}_1 &= x_2^2 - 8x_1 \\ \dot{x}_2 &= 2x_2 - x_1 x_2 \end{aligned}$$

does not have any periodic orbit. State any theorem that you invoke.

MAY 2013. **PRELIMINARY EXAMINATION**
Partial Differential Equations

Do three out of four problems below. Write in the following boxes the three problems that are to be graded:

Failing to clearly indicate three problems will result in Problems 1, 2 and 3 being graded.

1. Let $D = [-\pi, \pi] \times [a, b]$, where $a < b$ are two numbers. Let U be an open, connected subset of $D \times [0, \infty)$.

Let $t_0 > 0$ be fixed. Denote by Γ the parabolic boundary of U. We write $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset [t_0, \infty) \times D$ and $\Gamma_2 \subset [0, t_0) \times D$.

Suppose function $u \in C^2(U) \cap C(\bar{U})$ satisfies

$$\begin{aligned} \beta u_t - \Delta u &\le 0 \quad \text{in } U, \\ u(x,t) &\le 0 \quad \text{on } \Gamma_1, \\ u(x,t) &\le M \quad \text{on } \Gamma_2. \end{aligned}$$

Here, β is a positive constant and M is a constant.

Prove that there exist constants $A > 0$ and $C > 0$ such that

$$u(x,t) \le Ce^{-At} \quad \text{for all } (x,t) \in U. \tag{1}$$

2. Let U be an unbounded domain (open, connected set) in $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 > x_1 > 0\}$.

Suppose $u_1, u_2 \in C^2(U) \cap C(\bar{U})$ are two uniformly bounded functions in U that satisfy

$$\begin{cases} \Delta u_1 = \Delta u_2 & \text{in } U, \\ u_1 = u_2 & \text{on } \partial U, \end{cases}$$

Prove that $u_1 = u_2$ in U.

3. Let D be a bounded domain (open, connected set) in $\mathbb{R}^n$ and $U = D \times (0, \infty)$. Let $u = u(x,t) \in C^2(\bar{U})$ be a solution to the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = -k\dfrac{\partial u}{\partial t} & \text{in } U, \\ u(x,0) = g_0(x), \quad u_t(x,0) = g_1(x) & \text{on } D, \\ u(x,t) = 0, & \text{on } \partial D \times [0,\infty), \end{cases}$$

where $k > 0$ is a constant and g_0, g_1 are given functions.

Prove that

$$\lim_{t\to\infty} \int_D u^2(x,t)dx = 0.$$

Hints: (a) You may need to consider the following two functionals

$$E(t) = \frac{1}{2}\int_D (ku + u_t)^2 dx \quad \textit{and} \quad I(t) = \frac{1}{2}\int_D u_t^2 dx.$$

(b) You may need to use the following Poincaré's inequality without proof. There exists a positive constant C_p such that for any function $v(x) \in C^2(\bar{D})$ vanishing on the boundary ∂D, one has

$$\int_D |\nabla v|^2 dx \geq C_p \int_D v^2 dx.$$

4. Consider the Laplace equation

$$\Delta u = 0.$$

a) State a mean value formula.

b) State Harnack's inequality for positive solutions of the Laplace equation.

c) Prove the above Harnack's inequality using a mean value formula.