August 2014 Ph.D. Preliminary Exam in ODE

Solve all 4 problems. All problems have equal weight. However, subproblems may not have equal weight.

1. Let $\phi : [a,b] \to \mathbb{R}$ be a continuous function and $\mu : [a,b] \to \mathbb{R}$ be continuous and non-negative. Prove that, if a continuous function $y : [a,b] \to \mathbb{R}$ satisfies

$$\forall\, t \in [a,b], \qquad y(t) \le \phi(t) + \int_a^t \mu(s)\, y(s)\, ds,$$

then,

$$\forall\, t \in [a,b], \qquad y(t) \le \phi(t) + \int_a^t \phi(s)\, \mu(s)\, \exp\left[\int_s^t \mu(\tau)\, d\tau\right] ds.$$

2. Suppose A is a real $n \times n$ matrix.
   (a) Prove that if A is symmetric, then e^{At} is positive definite $\forall\, t \in \mathbb{R}$.
   (b) Prove that if A is skew-symmetric, e^{At} is an orthogonal matrix $\forall\, t \in \mathbb{R}$. *Hint:* An orthogonal matrix U satisfies $U\,U^T = I$.

3. Consider the system:

$$\begin{aligned} \dot{x} &= -x + y^2, \\ \dot{y} &= y + x^2. \end{aligned}$$

   (a) Find all the critical points for the system.
   (b) Find the stable eigenspace E^s and the unstable eigenspace E^u for the critical point $(0,0)$.
   (c) Construct successive approximate solutions $(x_i(t), y_i(t))$, $i = 0, 1, 2$ to find approximations to the stable and unstable manifolds at $(0,0)$.

4. Consider the system:

$$\dot{x} = \sin(2y) - \sin^3(2x); \qquad \dot{y} = -\sin(2x).$$

   (a) Find all the critical points for the system.
   (b) State La Salle's invariance principle. Determine the stability properties of the origin.
   (c) State Chetaev's instability theorem. Show that the point $(0, \frac{\pi}{2})$ is unstable using this theorem.

August 2014. **PRELIMINARY EXAMINATION**

Partial Differential Equations

Do three out of four problems below. Write in the following boxes the three problems that have to be graded, otherwise problems 1, 2, and 3 will be used for grading:

1. Let D be a bounded domain in $\mathbb{R}^n$, $U = D \times (t_0, T]$, and $\Gamma(U)$ be the parabolic boundary of U.

Define the operator

$$Lu = \sum_{i=1}^{n} \Big(a_i(x,t) u_{x_i x_i}(x,t) u + b_i(x,t) u_{x_i} \Big) - \alpha(x,t) u_t,$$

where $\alpha(x,t)$ and $a_i(x,t)$ are continuous, positive functions on $\overline{U}$ and $b_i(x,t)$ are bounded functions.

(a) State and prove the maximum principle for a classical solution of the differential inequality

$$Lu \leq 0.$$

(b) Prove that there exists at most one solution $u \in C^2(U) \cap C(\overline{U})$ of the following problem:

$$Lu = f(x,t) \quad \text{in} \quad U,$$

$$u(x,t) = g(x,t) \quad \text{on} \quad \Gamma(U).$$

2. Let U be a bounded domain in $\mathbb{R}^n$. Assume $u \in C^2(\overline{U})$ is classical solution of the problem

$$\Delta u + |\nabla u|^2 = 0 \quad \text{in} \quad U. \tag{1}$$

Let x_0 be a point in U. Prove that if $u(x_0) = \max_{\overline{U}} u(x)$ then $u(x) \equiv const.$

(Hint: Find a transformation $v = F(u)$ to reduce equation (1) to the Laplace equation for function v.)

3. Let $U = D \times [t_0, \infty)$, where D is a bounded domain in $\mathbb{R}^n$ with a smooth boundary. Let $a_{ij}(x)$ $(i, j = 1, \ldots, n)$ be functions on D that satisfy $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$, and there exist positive numbers α_0 and α_1 such that

$$\alpha_0 \sum_{i=1}^{n} \xi_i^2 \geq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha_1 \sum_{i=1}^{n} \xi_i^2 \quad \text{for all} \quad x \in D, \text{ and } \xi_1, \xi_2, \ldots, \xi_n \in \mathbb{R}.$$

Let $u_k = u_k(x,t) \in C^2(\overline{U})$, for $k = 1, 2$, be solutions of the problems

$$\sum_{i,j=1}^{n} (a_{ij}(x)(u_k)_{x_j})_{x_i} - u_{tt} = f(x,t) \quad \text{in} \quad U,$$

$$u_k(x, t_0) = g_k(x), \quad (u_k)_t(x, t_0) = h_k(x) \quad \text{on} \quad D,$$

$$u_k(x,t) = F(x,t) \quad \text{on} \quad \partial D \times [t_0, \infty),$$

where f and F are given functions, $h_1, h_2 \in C(D)$, and $g_1, g_2 \in C^1(D)$.

Let $w = u_1 - u_2$, $G = g_1 - g_2$, and $H = h_1 - h_2$.

Prove for $t > t_0$ that

(a)

$$\int_D \Big[w_t^2(x,t) + \sum_{i,j=1}^{n} \Big(a_{ij}(x) w_{x_j}(x,t) w_{x_i}(x,t) \Big) \Big] dx \equiv const,$$

(b)

$$\int_D (w_t^2(x,t) + |\nabla w(x,t)|^2) dx \leq C \int_D (H^2(x) + |\nabla G(x)|^2) dx,$$

for some constant $C > 0$.

4. Let $U = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0\}$. Let $u_1(x)$ and $u_2(x)$ be two uniformly bounded functions in $C^2(\overline{U})$. Assume that

$$\Delta u_1 = \Delta u_2 \quad \text{in } U, \quad \text{and} \quad u_1 = u_2 \quad \text{on } \partial U.$$

Prove that $u_1(x) = u_2(x)$ on U.