May 2014 Ph.D. Preliminary Exam in ODE

Solve all 4 problems. All problems have equal credit. However, subproblems may not have equal credit.

1. Consider the initial value problem (IVP):

$$\dot{x}(t) = f(t, x(t)) + g(t); \quad x(t_0) = x_0,$$

where $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfies:

$$\forall\, t \in \mathbb{R} \text{ and } x_1,\, x_2 \in \mathbb{R}^2, \qquad \|f(t, x_1) - f(t, x_2)\| \leq K(t)\, \|x_1 - x_2\|,$$

where $K : \mathbb{R} \to \mathbb{R}_+$ and $g : \mathbb{R} \to \mathbb{R}^2$ are continuous functions. Assume that solutions exist on the interval $[t_0, t_f]$ for all initial conditions x_0 and continuous functions g.

Let $z_1(\cdot)$ and $z_2(\cdot)$ be two solutions corresponding to initial conditions z_{01} and z_{02}, and continuous functions $g(t) = g_1(t)$ and $g(t) = g_2(t)$ respectively, on the interval $[t_0, t_f]$.

(i) Prove that

$$\forall\, t \in [t_0, t_f] \qquad \|z_1(t) - z_2(t)\| \leq (\|z_{01} - z_{02}\| + M)\, \exp\left[\int_{t_0}^{t} |K(s)|\, ds\right]$$

for some $M \geq 0$.

(ii) Prove that the initial value problem (IVP) has a unique solution on the interval $[t_0, t_f]$.

2. Consider the system:

$$\dot{x}(t) = A\, x(t) + B(t)\, x(t), \quad x(0) = x_0,$$

where, $\forall\, t \in [0, \infty)$, $\displaystyle\int_0^t \|B(s)\|\, ds < \infty$. Suppose A is a $n \times n$ real Hurwitz matrix, that is, for some $K, \lambda > 0$, and $\forall t \in [0, \infty)$,

$$\|e^{At}\| \leq K\, e^{-\lambda t}.$$

(i) Prove that the solution $x(t)$ of the system satisfies:

$$\forall\, t \in [0, \infty) \qquad e^{\lambda t}\, \|x(t)\| \leq K\, \|x_0\|\, \exp\left[K \int_0^t \|B(s)\|\, ds\right].$$

Hint: Apply the variation of parameters formula to the system.

(ii) Prove that the origin is exponentially stable.

3. Consider the system:

$$\begin{aligned}\dot{x} &= -x + y^2 \\ \dot{y} &= 2y + xy.\end{aligned}$$

(a) Find all the critical points for the system.

(b) Find the stable eigenspace E^s and the unstable eigenspace E^u for the critical point $(0, 0)$.

(c) Construct successive approximate solutions $(x_i(t), y_i(t))$, $i = 0, 1, 2$ to find approximations to the stable and unstable manifolds at $(0, 0)$.

4.

(a) Consider the system:

$$\begin{aligned}\dot{x}_1 &= x_2\,(1 - x_1^2) \\ \dot{x}_2 &= -(x_1 + x_2)\,(1 - x_1^2).\end{aligned}$$

Determine the stability properties of the origin. Provide a statement of any theorems you use to support your conclusion.

(b) Consider the system:

$$\begin{aligned}\dot{x}_1 &= x_1^2(x_1 + x_2) \\ \dot{x}_2 &= -x_2 + x_2^3 + x_1 x_2.\end{aligned}$$

Determine the stability properties of the origin. Provide a statement of any theorems you use to support your conclusion.

MAY 2014. **PRELIMINARY EXAMINATION**

Partial Differential Equations

Do three out of four problems below. Clearly indicate which of three problems have to be graded. Write in the following boxes the three problems that have to be graded, otherwise problems 1, 2, and 3 will be used for grading:

1. Let U be a bounded domain in $[t_0, T) \times \mathbb{R}^n$, and $\Gamma(U)$ be the parabolic boundary of U.

Define the operator

$$Lu = \alpha(x,t)\Delta u - \beta(x,t)u_t,$$

Here:

Functions $\alpha(x,t)$ is defined on $\overline{U}$ and satisfies

$$\alpha_0 < \alpha(x,t) < \alpha_0^{-1} \quad \text{in } U, \quad \text{for some constant } \alpha_0 > 0,$$

and

$$\beta_0 < \beta(x,t) < \beta_0^{-1} \quad \text{in } U, \quad \text{for some constant } \beta_0 > 0.$$

(a) State and prove the maximum principle for the classical solution of the differential inequality

$$Lu \leq 0$$

.

(b) Suppose $u \in C^2(U) \cap C(\overline{U})$ is a solution of

$$Lu = f(x,t) \quad \text{in } U,$$

$$u(x,t) = g(x,t) \quad \text{on } \Gamma(U).$$

Assume that

$$|f(x,t)| \leq K < \infty \quad \text{for all } (x,t) \in \overline{U},$$

and

$$|g(x,t)| \leq m < \infty \quad \text{for all } (x,t) \in \Gamma(U).$$

Prove that for some constant A

$$|u(x,t)| \leq m + A(t - t_0) \quad \text{for all } (x,t) \in U.$$

2. Let U be a bounded domain in $\mathbb{R}^n$.

(a) State without proof the mean value theorem in a ball $B(x_0, r) \subset U$ for solutions of the Laplace equation.

(b) Assume μ is a locally integrable function on $\mathbb{R}$, and $u \in C^2(U)$ is a bounded classical solution of the problem

$$\Delta u + \mu(u)|\nabla u|^2 = 0 \quad \text{in } U.$$

Prove that the function

$$w(x) = \int_0^{u(x)} \left(\exp \int_0^s \mu(\tau) d\tau \right) ds, \quad \text{for } x \in U, \tag{1}$$

has the mean value property.

3. Let $U = [t_0, \infty) \times D$, where D is a bounded domain in $\mathbb{R}^n$.

Prove that there is at most one solution $u = u(x,t) \in C^2(\bar{U})$ of the problem

$$\frac{\partial^2 u}{\partial t^2} + k(x,t)\frac{\partial u}{\partial t} = \Delta u \quad \text{in } U,$$

$$u(x, t_0) = g_0(x), \quad u_t(x, t_0) = g_1(x) \quad \text{on } D,$$

$$\frac{\partial u}{\partial \nu} = h(x,t) \quad \text{on } \partial D \times [t_0, \infty),$$

where $k(x,t) \geq 0$, $g_0(x)$, $g_1(x)$, and $h(x,t)$ are given and continuous functions.

4. Let $U = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0\}$. Let $u(x) \in C^2(\overline{U})$ be a bounded solution of the problem

$$\Delta u = 0 \quad \text{in } U, \quad \text{and} \quad \frac{\partial u}{\partial x_1} = 0 \quad \text{on } \partial U.$$

Prove that u is a constant. (*Hint: You can use the method of reflection.*)