2015 August ODE/PDE Preliminary Examination

Part I: ODE. Do 3 of the following 4 problems. You must clearly indicate which 3 are to be graded. Problems 1, 2, and 3 will be graded if no indication is given. Strive for clear and detailed solutions.

1. Find a fundamental matrix for the time-variant system

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_3 \\ \dot{x}_2 &= -2x_2 + 2tx_3 \\ \dot{x}_3 &= x_3 \end{aligned}$$

2. Let $E \subset \mathbb{R}^n$ be open and $f : E \mapsto \mathbb{R}^n$ be continuously differentiable on E. Prove that f is locally Lipschitz on E.

3. Investigate the stability of the origin of the system

$$\begin{aligned} \dot{x}_1 &= -2x_1 - 3x_2 + 2x_3^2 + x_1^2 - 2x_1x_3 \\ \dot{x}_2 &= x_1 + x_2 \\ \dot{x}_3 &= x_3^2 + x_1^2 - 2x_1x_3 \end{aligned}$$

   by using the center manifold theorem.

4. Show that the following systems have no periodic orbit

   a)

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= 2x_1 + x_2 + (x_1 + x_2)(x_1 - x_2) \end{aligned}$$

   b)

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^3, \\ \dot{x}_2 &= x_2(1 - x_1^2) \end{aligned}$$

AUGUST 2015. **PRELIMINARY EXAMINATION**

Partial Differential Equations

Do three out of four problems below. Clearly indicate in the following boxes which three problems to be graded; otherwise problems 1, 2, and 3 will be used for grading:

1. Let D be a bounded domain in $\mathbb{R}^n$ with C^1-boundary, and $U = D \times (0, \infty)$. Let Γ denote the parabolic boundary of U.

Let α_1, α_2 be two positive numbers.

Let $u_1(x,t)$ and $u_2(x,t)$ with $(x,t) \in \bar{U}$ be two functions in $C^2(\bar{U})$.

Suppose

$$\frac{\partial u_1}{\partial t} - \alpha_1 \Delta u_1 = \frac{\partial u_2}{\partial t} - \alpha_2 \Delta u_2 \quad \text{on } U,$$

and

$$u_1 = u_2 \quad \text{on } \Gamma.$$

Assume there is $M > 0$ such that

$$\int_D \Big(|\nabla u_1(x,t)|^2 + |\nabla u_2(x,t)|^2 \Big) dx \le M \quad \forall t > 0.$$

Prove that there exists a constant $C > 0$ such that

$$\int_D |u_1(x,t) - u_2(x,t)|^2 dx \le C |\alpha_1 - \alpha_2|^2 \quad \forall t \ge 0.$$

(Note: Poincaré's inequality can be used without proof.)

2. Let D be a bounded domain in $\mathbb{R}^n$ and $U = D \times (0, \infty)$. Let Γ denote the parabolic boundary of U. Suppose $u(x,t)$ is a classical solution of the problem

$$u_t(x,t) - \Delta u(x,t) = 1 \quad \forall (x,t) \in U,$$

and

$$u(x,t) = 1 \quad \forall (x,t) \in \Gamma.$$

Prove that for any $\alpha > 1$ and $x \in D$ one has

$$\lim_{t \to \infty} \left(\frac{u(x,t)}{t^\alpha} \right) = 0.$$

3. Let $u(x,t)$ be the classical solution of the wave equation

$$u_{tt}(x,t) - u_{xx}(x,t) = 0 \quad \text{in} \quad \mathbb{R} \times (0,\infty)$$

satisfying the initial conditions

$$u(x,0) = g(x) \quad \text{and} \quad u_t(x,0) = h(x),$$

where $g(x)$ and $h(x)$ are given functions on $\mathbb{R}$.

Assume

$$\lim_{x\to\pm\infty} g(x) = 0,$$

$$h(x) \geq 0 \quad \forall x \in \mathbb{R} \quad \text{and} \quad \int_{-\infty}^{+\infty} h(x)dx < \infty.$$

Prove for any fixed $t > 0$ that

$$\lim_{x\to\pm\infty} u(x,t) = 0.$$

4. Suppose $u(x)$ is a classical, bounded solution of the Laplace equation in $\mathbb{R}^n$, that is, $u \in C^2(\mathbb{R}^n)$, $|u(x)| \leq C$ and $\Delta u(x) = 0$ for all $x \in \mathbb{R}^n$, where C is a positive constant.

Prove that $u = \textit{constant}$ on $\mathbb{R}^n$.